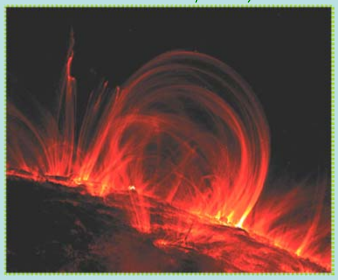
ASTR 7500: Solar & Stellar Magnetism

Hale CGEG Solar & Space Physics



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Lecture 8 Thurs 14 Jan 2013

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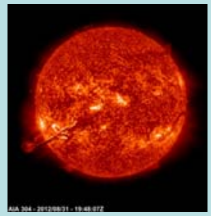
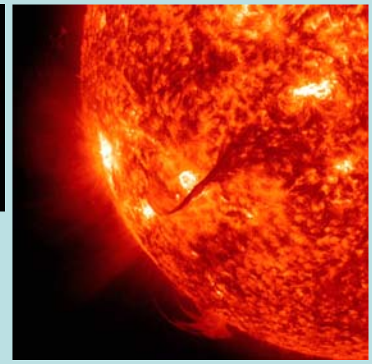
Magnetic Eruption in AIA 304 Å



On August 31st, a C-class solar flare caused a filament eruption. The magnetic structure hit Earth with a glancing blow provoking lovely auroral displays on Sep 3.



REMINDER MHD Equations

The kingpin equation is the momentum equation (Navier-Stokes augmented by gravity and the Lorentz force)

Momentum Equation
$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} P + \rho \vec{g} + \mu \left[\nabla^2 \vec{v} + \frac{1}{3} \vec{\nabla}\left(\vec{\nabla} \cdot \vec{v}\right) \right] + \frac{1}{c} \vec{J} \times \vec{B}$$

Evolution equations for the secondary variables

Mass Continuity
$$\frac{D\rho}{Dt} = -\rho \vec{\nabla} \cdot \vec{v}$$

Internal Energy Equation
$$c_v \frac{DT}{Dt} = -(\gamma - 1) c_v T \vec{\nabla} \cdot \vec{v} + Q$$

Magnetic Induction
$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B}\right) + \eta \nabla^2 \vec{B}$$

Supplemental Equations

Advective Derivative
$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}$$

Nonadiabatic Terms
$$Q = Q_{\rm visc} + Q_{\rm ohm} + Q_{\rm diff} + \cdots$$

Maxwell's Equations
$$\vec{\nabla} \cdot \vec{B} = 0$$ Solenoidal Constraint
$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}$$ Ampère's Law (pre-Maxwell)

Equation of State
$$P = \rho R T$$

3

REMINDER Induction Equation

Eliminate all variables but the fluid velocity and the magnetic field. Use Ohm's Law to eliminate the electric field. Use Ampère's Law to eliminate the current density.

Faraday's Law
$$\frac{\partial \vec{B}}{\partial t} = -c \vec{\nabla} \times \vec{E}$$

Ohm's Law
$$\vec{J} = \sigma \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right)$$
$$\vec{E} = -\frac{\vec{v}}{c} \times \vec{B} + \frac{\vec{J}}{\sigma}$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B}\right) - \vec{\nabla} \times \left(\frac{c}{\sigma} \vec{J} \right)$$

Ampère's Law
$$\vec{J} = \frac{c}{4\pi} \vec{\nabla} \times \vec{B}$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B}\right) - \vec{\nabla} \times \left(\frac{c^2}{4\pi\sigma} \vec{\nabla} \times \vec{B} \right)$$

4

REMINDER Magnetic Diffusivity

The quantity $c^2/4\pi\sigma$ is called the magnetic diffusivity and the term containing it represents the diffusion of magnetic field

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B}\right) - \vec{\nabla} \times \left(\eta \vec{\nabla} \times \vec{B} \right)$$

$$\eta = \frac{c^2}{4\pi\sigma}$$ Magnetic Diffusivity

The diffusive nature of this term can be easily recognized if we consider a constant diffusivity

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B}\right) + \eta \nabla^2 \vec{B}$$

where I have used
$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B} \right) = -\nabla^2 \vec{B} + \vec{\nabla}\left(\vec{\nabla} \cdot \vec{B} \right)$$
(Solenoidal Constraint: $\vec{\nabla} \cdot \vec{B} = 0$)

5

REMINDER Vorticity Analogy

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B}\right) + \eta \nabla^2 \vec{B}$$

This equation is identical to the vorticity equation for barotropic fluids $\rho = \rho(P)$,

$$\frac{\partial \vec{\omega}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{\omega}\right) + \nu \nabla^2 \vec{\omega}$$

$$\vec{\omega} = \vec{\nabla} \times \vec{v}$$ Vorticity

$$\nu = \frac{\mu}{\rho}$$ Kinematic viscosity

Therefore, the magnetic field evolves in the same fashion that vortex lines do.

6

Induction Equation

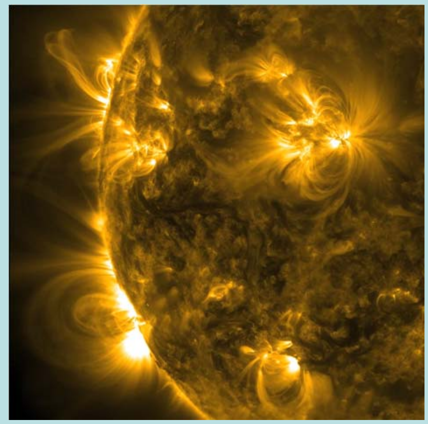
PRIMARY GOAL

Magnetic Diffusion

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right) + \eta \nabla^2 \vec{B}$$

Flux Freezing
(the field moves with the fluid)

7



8

Example: Decay of the Interstellar Field

Let's calculate the time scale for the decay of the interstellar magnetic field. The relevant equation is given by the diffusive limit of the induction equation

$$\frac{\partial \vec{B}}{\partial t} = \eta \nabla^2 \vec{B} \longrightarrow \tau = \frac{L^2}{\eta} \quad \text{Time scale}$$

With the parameters

$$L = 1 \text{ parsec} = 3 \times 10^{18} \text{ cm}$$

$$\sigma \approx 2 \times 10^7 \;\; T^{1.5} \;\; \text{s}^{-1}$$

$$\eta = \frac{c^2}{4\pi\sigma} \approx 3.6 \times 10^6 \;\; \text{cm}^2 \;\; \text{s}^{-1}$$

$$\tau \approx 8 \times 10^{22} \text{ years}$$

The decay time for the interstellar magnetic field is much longer than the lifetime of the universe (10^{10} years).

9

The Lorentz Force

10

Momentum Equation

If we augment the Navier-Stokes equation with gravitational, electric and magnetic forces we obtain

Small and ignorable (non-relativistic)

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} P + \rho \vec{g} + \vec{\nabla} \cdot \vec{\tau} + q\vec{E} + \frac{1}{c} \vec{J} \times \vec{B}$$

Pressure Force

Gravitational Force

Viscous Force

$$\vec{\nabla} \cdot \vec{\tau} \equiv \mu \left[\nabla^2 \vec{v} + \frac{1}{3} \vec{\nabla} \left(\vec{\nabla} \cdot \vec{v} \right) \right]$$

Electric Force

Magnetic Force

Lorentz Force

11

Magnetic Forces

We can use Ampère's Law to eliminate the current density

$$\rho \left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \right) \vec{v} = -\vec{\nabla} P + \rho \vec{g} + \vec{\nabla} \cdot \vec{\tau} + \frac{1}{c} \vec{J} \times \vec{B}$$

Ampère's Law

$$\vec{J} = \frac{c}{4\pi} \vec{\nabla} \times \vec{B}$$

$$\rho \left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \right) \vec{v} = -\vec{\nabla} P + \frac{1}{4\pi} \left(\vec{\nabla} \times \vec{B} \right) \times \vec{B} + \rho \vec{g} + \vec{\nabla} \cdot \vec{\tau}$$

12

MHD Momentum Equation

This form of the momentum equation is important enough to get its own slide

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} P + \frac{1}{4\pi}\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B} + \rho\vec{g} + \vec{\nabla}\cdot\overleftrightarrow{\tau}$$

Pressure Force — Lorentz Force — Gravitational Force — Viscous Stress

Magnetic Pressure and Tension

Magnetic Tension and Pressure

A common form for the Lorentz force can be obtained by using the following vector identity

$$\vec{\nabla}\left(\vec{a}\cdot\vec{b}\right) = (\vec{a}\cdot\vec{\nabla})\vec{b} + (\vec{b}\cdot\vec{\nabla})\vec{a} + \vec{a}\times\left(\vec{\nabla}\times\vec{b}\right) + \vec{b}\times\left(\vec{\nabla}\times\vec{a}\right)$$

or with $\vec{a} = \vec{b} = \vec{B}$

$$\vec{\nabla}\left(\vec{B}\cdot\vec{B}\right) = 2(\vec{B}\cdot\vec{\nabla})\vec{B} + 2\vec{B}\times\left(\vec{\nabla}\times\vec{B}\right)$$

The Lorentz force becomes

$$\frac{1}{4\pi}\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B} = -\vec{\nabla}\left(\frac{B^2}{8\pi}\right) + \frac{(\vec{B}\cdot\vec{\nabla})\vec{B}}{4\pi}$$

Magnetic Pressure Force — Magnetic Tension Force

Full MHD Momentum Equation

$$\rho\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\vec{\nabla}\left(P + \frac{B^2}{8\pi}\right) + \frac{(\vec{B}\cdot\vec{\nabla})\vec{B}}{4\pi} + \rho\vec{g} + \mu\nabla^2\vec{v}$$

Magnetic Pressure — Magnetic Tension — Total Pressure

Magnetic Pressure

Magnetic pressure forces arise from gradients of field strength.
Magnetic pressure is isotropic.

Magnetic Tension

Magnetic tension forces arise from curvature of field lines.
Magnetic tension depends on the direction of the field.

Magnetic Tension

Guitar String Analogy

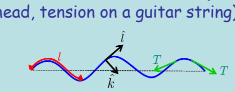
Magnetic tension is analogous to other tension forces (surface tension, tension on a drum head, tension on a guitar string).

$$\rho\frac{d\vec{v}}{dt} = \vec{\nabla}\cdot\vec{T}$$

$$= \vec{\nabla}\cdot\left(T\hat{l}\right)$$

$$\rho\frac{d\vec{v}}{dt} = T\frac{\partial\hat{l}}{\partial l} = T\frac{\hat{k}}{R}$$

$$\rho\frac{d\vec{v}}{dt} = T\frac{\hat{k}}{R}$$



T - Tension

$\hat{l}$ - Tangent direction

$\hat{k}$ - Principle normal direction

R - Radius of curvature

$$\frac{\partial\hat{l}}{\partial l} \equiv \frac{\hat{k}}{R}$$

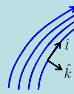
The tension pulls along the guitar string (in the tangential direction), but the net force is in the normal direction.

Parallel-Transverse Decomposition

For simplicity, consider a 2D field. Decompose the forces into components that are parallel and transverse to the magnetic field. The parallel (or tangent direction) is $\hat{l}$ while the transverse (or principle normal direction) is $\hat{k}$.

$$\rho\frac{D\vec{v}}{Dt} = -\vec{\nabla}\left(\frac{B^2}{8\pi}\right) + \frac{(\vec{B}\cdot\vec{\nabla})\vec{B}}{4\pi}$$

$$\rho\frac{D\vec{v}}{Dt} = -\frac{\partial}{\partial l}\left(\frac{B^2}{8\pi}\right)\hat{l} - \frac{\partial}{\partial k}\left(\frac{B^2}{8\pi}\right)\hat{k} + \frac{1}{4\pi}B\frac{\partial(B\hat{l})}{\partial l}$$



$\vec{B} = B\hat{l}$

$$= -\frac{\partial}{\partial l}\left(\frac{B^2}{8\pi}\right)\hat{l} - \frac{\partial}{\partial k}\left(\frac{B^2}{8\pi}\right)\hat{k} + \frac{1}{8\pi}\frac{\partial B^2}{\partial l}\hat{l} + \frac{B^2}{4\pi}\frac{\partial\hat{l}}{\partial l}$$

$$\rho\frac{D\vec{v}}{Dt} = -\frac{\partial}{\partial k}\left(\frac{B^2}{8\pi}\right)\hat{k} + \frac{B^2}{4\pi}\frac{\hat{k}}{R}$$

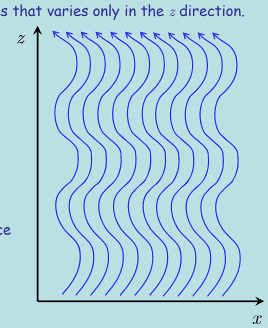
Magnetic Tension $T = \frac{B^2}{4\pi}$

19

Tension and Curvature

Consider a field with sinusoidal field lines that varies only in the z direction.

$$B_x = b\cos(kz)$$
$$B_y = 0$$
$$B_z = B_0$$



For small b

$$B^2 = B_0^2 + \mathcal{O}\left(b^2\right)$$

Therefore, the magnetic pressure force vanishes (to linear order in b)

$$-\vec{\nabla}\left(\frac{B^2}{8\pi}\right) = \mathcal{O}\left(b^2\right)$$

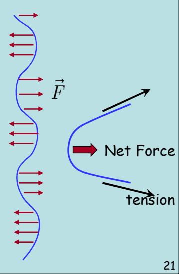
20

The Tension

To linear order in b, there is a nonzero tension force, however.

$$\frac{(\vec{B}\cdot\vec{\nabla})\vec{B}}{4\pi} = \frac{B_0}{4\pi}\frac{\partial\vec{B}}{\partial z} = -\frac{bB_0k}{4\pi}\sin(kz)\,\hat{x} + \mathcal{O}\left(b^2\right)$$

$$\frac{(\vec{B}\cdot\vec{\nabla})\vec{B}}{4\pi} \approx -\left[\frac{bB_0k}{4\pi}\sin(kz)\right]\hat{x}$$



Remember that $B_x = b\cos(kz)$

Therefore, the tension force is 90 degrees out of phase with B_x

The tension force tries to straighten the field line

21

Consequences of Magnetic Pressure

Magnetic Evacuation and Buoyancy

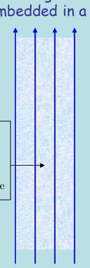
22

Evacuation of a Flux Tube

Example: A Static Flux Tube

Imagine a straight tube of magnetic flux in equilibrium with its surroundings. The tube has a field strength of B, and the tube is embedded in a nonmagnetized fluid with a pressure of P_{ext}.

Inside
$\vec{B} = B\hat{z}$
$P = P_{tube}$



Outside
$\vec{B} = 0$
$P = P_{ext}$

Since the tube is static, the gas pressure inside the tube can be determined by the requirement that the gradient of the total pressure vanishes.

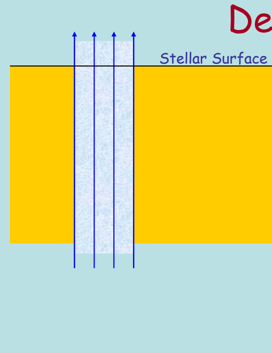
In other words the total pressure (gas + magnetic) is constant.

$$P_{tube} + \frac{B^2}{8\pi} = P_{ext}$$

$$P_{tube} = P_{ext} - \frac{B^2}{8\pi}$$

23

Reduced Gas Pressure and Density



Since the pressure (gas plus magnetic) must be the same inside and outside the tube

$$P_{tube} = P_{ext} - \frac{B^2}{8\pi}$$

The gas pressure inside is less than outside

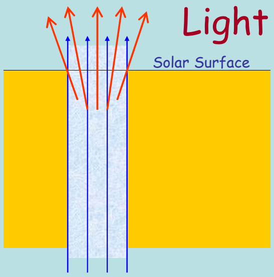
$$P_{tube} < P_{ext}$$

If the temperature is the same inside and outside (due to thermal diffusion), then the density is less inside than outside.

$$\rho_{tube} < \rho_{ext}$$

24

Light Fibers



Since the density inside a magnetic tube is less than the fluid outside, the tube's opacity is less

$$\chi_{\rm tube} < \chi_{\rm ext}$$

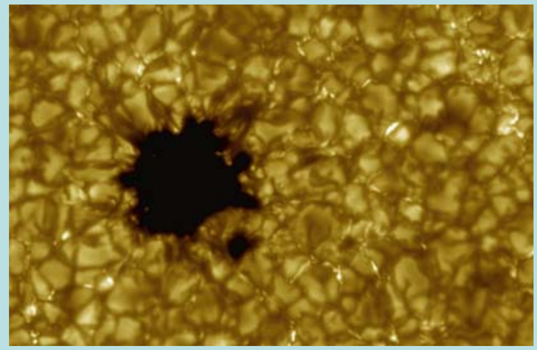
Thus the tube radiates more efficiently.

On the surface of the sun, large flux tubes (sunspots and pores) are dark because the fluid inside is cold (the magnetic field inhibits convection and the heat flux associated with the convective motions is low).

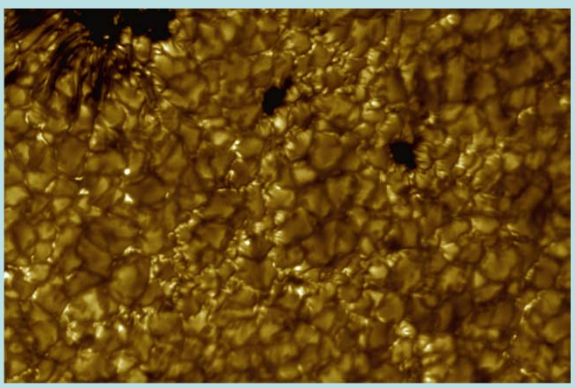
Small flux tubes are bright because they are partially evacuated with a reduced opacity.

25

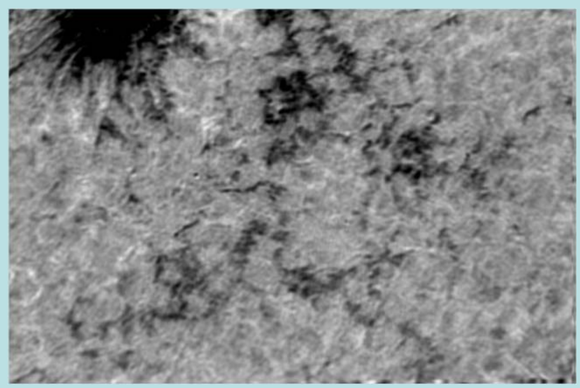
Pores and Bright Points



26



27

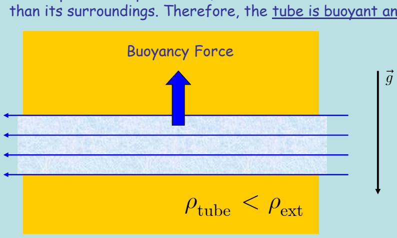


28

Magnetic Buoyancy

Imagine a horizontal flux tube embedded in a gravitationally stratified fluid (a star perhaps).

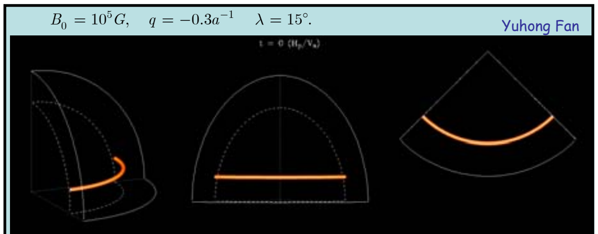
Due to pressure equilibration, the fluid inside the tube is less dense than its surroundings. Therefore, the tube is buoyant and should rise.



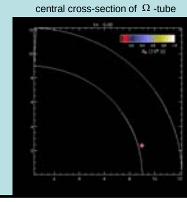
29

$B_0 = 10^5 G, \quad q = -0.3a^{-1} \quad \lambda = 15°.$

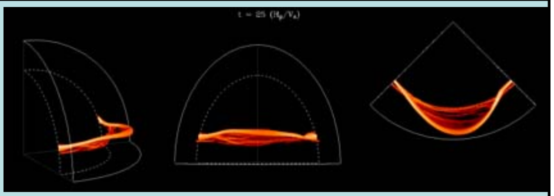
Yuhong Fan



central cross-section of Ω -tube



30



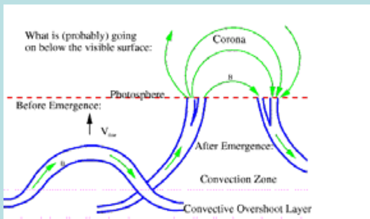
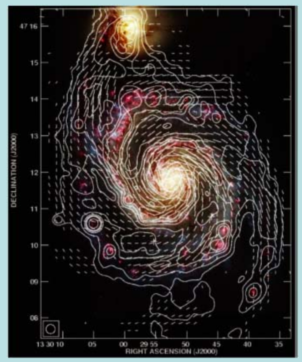


Figure by George Fisher

31

The Galactic Magnetic Field



Optical Map from Hubble

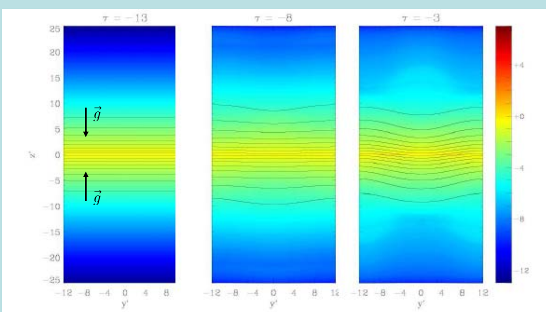
VLA Radio

Field deduced from radio polarization

M51 – Whirlpool Galaxy

32

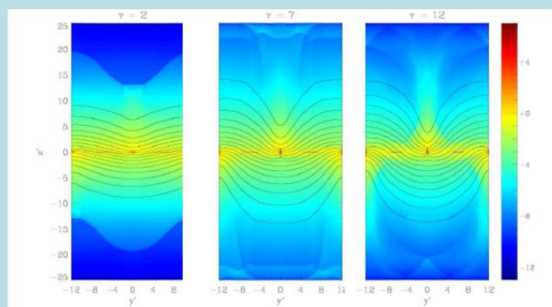
Formation of Interstellar Clouds



Telemachos Ch. Mouschovias

33

Parker Instability



34

Magnetic Energy

35

Derive an Energy Equation

The energy density of the magnetic field is equal to the magnetic pressure

$$W = \frac{B^2}{8\pi}$$

Compute the rate of change of the magnetic energy density

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) = \frac{1}{8\pi}\frac{\partial}{\partial t}\vec{B}\cdot\vec{B} = \frac{1}{4\pi}\vec{B}\cdot\frac{\partial \vec{B}}{\partial t}$$

We could use the induction equation to evaluate the right hand side; however, its more convenient to start with the root equation of Faraday's Law.

36

Use Faraday's Law

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) = \frac{1}{4\pi}\vec{B}\cdot\frac{\partial \vec{B}}{\partial t}$$

Faraday's Law

$$\frac{\partial \vec{B}}{\partial t} = -c\vec{\nabla}\times\vec{E}$$

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) = -\frac{c}{4\pi}\vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right)$$

$$\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right) = \vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right) - \vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right)$$

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) = -\frac{c}{4\pi}\left[\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right) + \vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right)\right]$$

37

Poynting Flux

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) = -\frac{c}{4\pi}\left[\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right) + \vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right)\right]$$

The first term is the divergence of the Poynting flux.

$$\vec{\Upsilon} = \frac{c}{4\pi}\left(\vec{E}\times\vec{B}\right)$$ Poynting Flux

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) = -\vec{\nabla}\cdot\vec{\Upsilon} - \frac{c}{4\pi}\vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right)$$

Ampère's Law

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}$$

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) = -\vec{\nabla}\cdot\vec{\Upsilon} - \left(\vec{E}\cdot\vec{J}\right)$$

38

Eliminate the Electric Field

Use the Lorentz transformation and Ohm's Law

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) = -\vec{\nabla}\cdot\vec{\Upsilon} - \left(\vec{E}\cdot\vec{J}\right)$$

Ohm's Law

$$\vec{E} = -\frac{\vec{v}}{c}\times\vec{B} + \frac{\vec{J}}{\sigma}$$

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) + \vec{\nabla}\cdot\vec{\Upsilon} = -\left(\frac{\vec{J}}{\sigma} - \frac{\vec{v}}{c}\times\vec{B}\right)\cdot\vec{J}$$

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = -\vec{b}\cdot\left(\vec{a}\times\vec{c}\right)$$

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) + \vec{\nabla}\cdot\vec{\Upsilon} = -\frac{J^2}{\sigma} - \vec{v}\cdot\left(\frac{\vec{J}\times\vec{B}}{c}\right)$$

39

Magnetic Energy Equation

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) + \vec{\nabla}\cdot\vec{\Upsilon} = -\frac{J^2}{\sigma} - \vec{v}\cdot\left(\frac{\vec{J}\times\vec{B}}{c}\right)$$

Poynting Flux

$$\vec{\Upsilon} = \frac{c}{4\pi}\left(\vec{E}\times\vec{B}\right)$$

Ohmic Dissipation

Rate of Work Performed on the Fluid

$$-\vec{v}\cdot\vec{F}_{\mathrm{Lorentz}}$$

$$c_v\frac{DT}{Dt} = -(\gamma-1)c_vT\,\vec{\nabla}\cdot\vec{v} + \sigma^{-1}J^2 + \cdots$$

If the resistivity, σ^{-1}, is large enough, one might need to include resistive heating (ohmic dissipation) in the internal energy equation.

40

Recap of MHD

Two Forms of Momentum Equation

$$\rho\frac{D\vec{v}}{Dt} = -\vec{\nabla}P + \frac{1}{4\pi}\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B} + \rho\vec{g} + \vec{\nabla}\cdot\overleftrightarrow{\tau}$$

$$\rho\frac{D\vec{v}}{Dt} = -\vec{\nabla}\left(P + \frac{B^2}{8\pi}\right) + \frac{(\vec{B}\cdot\vec{\nabla})\vec{B}}{4\pi} + \rho\vec{g} + \vec{\nabla}\cdot\overleftrightarrow{\tau}$$

Mass Continuity Equation

$$\frac{D\rho}{Dt} = -\rho\vec{\nabla}\cdot\vec{v}$$

Thermal Energy Equation

$$c_v\frac{DT}{Dt} = -(\gamma-1)c_vT\,\vec{\nabla}\cdot\vec{v} + Q$$

Magnetic Induction Equation

$$\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times\left(\vec{v}\times\vec{B}\right) + \eta\nabla^2\vec{B}$$

<u>Ohmic Heating</u>

$$Q_{\mathrm{ohm}} = \sigma^{-1}J^2$$

<u>Current Density</u>

$$\vec{J} = \frac{c}{4\pi}\vec{\nabla}\times\vec{B}$$

Magnetic Energy Equation

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) + \vec{\nabla}\cdot\vec{P} = -\frac{J^2}{\sigma} - \vec{v}\cdot\left(\frac{\vec{J}\times\vec{B}}{c}\right)$$

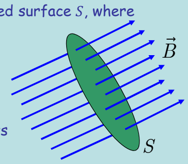
41

42

Alfvén's Theorem

Let Φ be the magnetic flux passing through a closed surface S, where the surface follows the fluid

$$\Phi = \int_S \vec{B} \cdot d\vec{s}$$



The magnetic flux can change in time t in two ways
(1) The magnetic field can evolve.
(2) The surface S can evolve.

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_S \vec{B} \cdot d\vec{s} = \dot{\Phi}_1 + \dot{\Phi}_2$$

$\dot{\Phi}_1$ - evolution of the field

$\dot{\Phi}_2$ - evolution of the surface

If the same field lines pass through the surface S for all time, the flux should be constant, $\dot{\Phi} = 0$.

Evolution of the Field

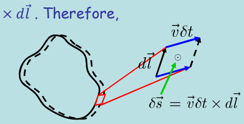
The rate of change due to the evolution of the field itself is simply

$$\dot{\Phi}_1 = \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

The change due to the evolution of the surface arises from the instantaneous motion of the boundary. Let ∂S be the boundary of the surface S. Over time δt, the boundary moves a distance $\vec{v}\delta t$. An element of the boundary, $d\vec{l}$ sweeps out an area $\vec{v}\delta t \times d\vec{l}$. Therefore,

$$\delta\Phi_2 = \oint_C \vec{B} \cdot \delta\vec{s}$$

$$\delta\Phi_2 = \oint_C \vec{B} \cdot \left(\vec{v}\delta t \times d\vec{l}\,\right)$$



Evolution of the Surface

Divide through by δt to obtain the rate of change.

$$\dot{\Phi}_2 = \lim_{\delta t \to 0} \frac{\delta\Phi_2}{\delta t} = \oint_C \vec{B} \cdot \left(\vec{v} \times d\vec{l}\,\right)$$

$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \times \vec{b}\right) \cdot \vec{c}$

$$\dot{\Phi}_2 = \oint_C \left(\vec{B} \times \vec{v}\right) \cdot d\vec{l}$$

Stokes Theorem

$$\dot{\Phi}_2 = \int_S \vec{\nabla} \times \left(\vec{B} \times \vec{v}\right) \cdot d\vec{s}$$

$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

$$\dot{\Phi}_2 = -\int_S \vec{\nabla} \times \left(\vec{v} \times \vec{B}\right) \cdot d\vec{s}$$

Frozen in Flux

Insert the two expressions into the rate of change of the flux

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_S \vec{B} \cdot d\vec{s} = \dot{\Phi}_1 + \dot{\Phi}_2$$

$$\frac{d\Phi}{dt} = \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} - \int_S \vec{\nabla} \times \left(\vec{v} \times \vec{B}\right) \cdot d\vec{s}$$

$$\frac{d\Phi}{dt} = \int_S \left[\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times \left(\vec{v} \times \vec{B}\right)\right] \cdot d\vec{s}$$

Since the expression in the square brackets is zero (due to the induction equation)

$$\frac{d\Phi}{dt} = 0$$

Voila!