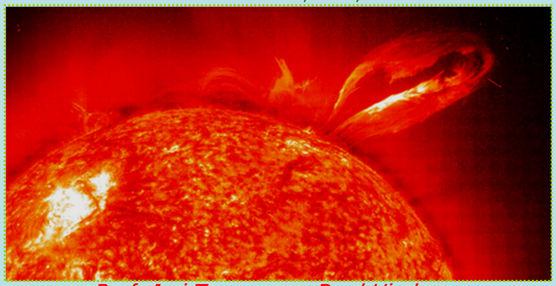
ASTR 7500: Solar & Stellar Magnetism

Hale CGEG Solar & Space Physics

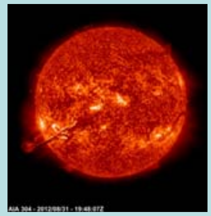


Prof. Juri Toomre + Brad Hindman

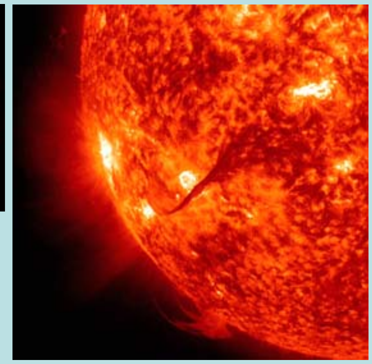
Lecture 7   Tues 12 Jan 2013

zeus.colorado.edu/astr7500-toomre

Magnetic Eruption in AIA 304 Å

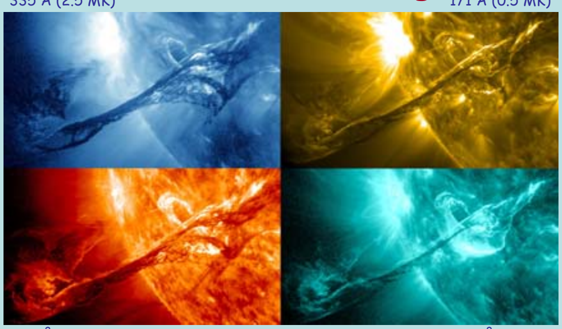


On August 31st, a C-class solar flare caused a filament eruption. The magnetic structure hit Earth with a glancing blow provoking lovely auroral displays on Sep 3.



Other AIA Wavelengths

335 Å (2.5 MK)   171 Å (0.5 MK)



304 Å (50 kK)   131 Å (10 MK)

Associated Aurora

Magnetohydrodynamics (MHD)

- The MHD Equations
- The Induction Equation
  - Magnetic Diffusivity
  - Ideal MHD
  - Alfvén's Theorem & Flux Freezing
- The Lorentz Force
  - Magnetic Pressure and Tension
  - Magnetic Stress Tensor
  - Magnetic Tension
  - Consequences of Magnetic Pressure (Magnetic Evacuation & Buoyancy)
- Magnetic Energy

5

MHD Equations

6

The Assumptions of MHD

Large Spatial and Temporal Scales

- The relevant scales must be large enough that we can treat the plasma as a single fluid and ignore electrostatic waves

  $L \gg \lambda_D$ *Spatial scales are much larger than the Debye length*

  $\tau \gg 1/\omega_p$ *Time scales are much larger than the inverse plasma frequency*

  (i.e., we can adopt a continuum, single fluid approximation)

Charge Neutrality

- This is a natural consequence of *Large Spatial and Temporal Scales* as long as the plasma conductivity is high.

Non-relativistic (non-essential, but convenient)

- We will ignore all terms that are proportional to v^2/c^2 and smaller.
- This approximation leads to the conclusion that electric fields and forces are small compared to magnetic fields and forces.

MHD Equations

The kingpin equation is the momentum equation (Navier-Stokes augmented by gravity and the Lorentz force)

Momentum Equation

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} P + \rho \vec{g} + \mu \left[\nabla^2 \vec{v} + \frac{1}{3} \vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right) \right] + \frac{1}{c} \vec{J} \times \vec{B}$$

Evolution equations for the secondary variables

Mass Continuity

$$\frac{D\rho}{Dt} = -\rho \vec{\nabla}\cdot\vec{v}$$

Internal Energy Equation

$$c_v \frac{DT}{Dt} = -(\gamma - 1) c_v T \vec{\nabla}\cdot\vec{v} + Q$$

Magnetic Induction

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times\left(\vec{v}\times\vec{B}\right) + \eta \nabla^2 \vec{B}$$

Supplemental Equations

Advective Derivative

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}$$

Nonadiabatic Terms

$$Q = Q_{\rm visc} + Q_{\rm ohm} + Q_{\rm diff} + \cdots$$

Maxwell's Equations

$$\vec{\nabla}\cdot\vec{B} = 0$$ Solenoidal Constraint

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}$$ Ampère's Law (pre-Maxwell)

Equation of State

$$P = \rho R T$$

Induction Equation

Magnetic Induction

The induction equation is really Faraday's Law from electrodynamics in disguise.

$$\frac{\partial \vec{B}}{\partial t} = -c \vec{\nabla}\times\vec{E}$$ Faraday's Law

Since the electric field is a passive variable in MHD (its tiny in comparison to the magnetic field), we would like to eliminate it. This is accomplished by using a generalized form of Ohm's Law appropriate for a moving fluid.

$$\vec{J} = \sigma\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$ Generalized Ohm's Law

σ - Conductivity

Ohm's Law in Electrostatics

$$\vec{J} = \sigma \vec{E}$$

This law should apply in a reference frame without plasma motion, i.e., in a reference frame co-moving with the fluid.

The Lorentz transformations tell us what the electric field is in this frame:

$$\vec{E}' = \vec{E} + \frac{\vec{v}}{c}\times\vec{B}$$

The Induction Equation

Eliminate all variables but the fluid velocity and the magnetic field. Use Ohm's Law to eliminate the electric field. Use Ampère's Law to eliminate the current density.

Faraday's Law

$$\frac{\partial \vec{B}}{\partial t} = -c \vec{\nabla}\times\vec{E}$$

Ohm's Law

$$\vec{J} = \sigma\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$

$$\vec{E} = -\frac{\vec{v}}{c}\times\vec{B} + \frac{\vec{J}}{\sigma}$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times\left(\vec{v}\times\vec{B}\right) - \vec{\nabla}\times\left(\frac{c}{\sigma}\vec{J}\right)$$

Ampère's Law

$$\vec{J} = \frac{c}{4\pi}\vec{\nabla}\times\vec{B}$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times\left(\vec{v}\times\vec{B}\right) - \vec{\nabla}\times\left(\frac{c^2}{4\pi\sigma}\vec{\nabla}\times\vec{B}\right)$$

Magnetic Diffusivity

The quantity $c^2/4\pi\sigma$ is called the magnetic diffusivity and the term containing it represents the diffusion of magnetic field

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times\left(\vec{v}\times\vec{B}\right) - \vec{\nabla}\times\left(\eta \vec{\nabla}\times\vec{B}\right)$$

$$\eta = \frac{c^2}{4\pi\sigma}$$ Magnetic Diffusivity

The diffusive nature of this term can be easily recognized if we consider a constant diffusivity

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times\left(\vec{v}\times\vec{B}\right) + \eta \nabla^2 \vec{B}$$

where I have used

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right) = -\nabla^2\vec{B} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{B}\right)$$

0 Solenoidal Constraint

What Do These Terms Represent?

Magnetic Diffusion

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right) + \eta \nabla^2 \vec{B}$$

What's this term?

13

Magnetic Reynolds Number

The relative importance of the two terms is given by the Magnetic Reynolds Number

$$t = \tau t'$$

$$\frac{\partial \vec{B}}{\partial t'} = \vec{\nabla}' \times \left(\vec{v}' \times \vec{B} \right) + \frac{1}{\mathcal{R}_M} \nabla'^2 \vec{B} \qquad \vec{x} = L \vec{x}'$$

$$\mathcal{R}_M = \frac{L^2}{\eta \tau}$$

The Magnetic Reynolds Number defines two regimes

$$\mathcal{R}_M \gg 1 \qquad \frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right)$$ Ideal MHD Limit

$$\mathcal{R}_M \ll 1 \qquad \frac{\partial \vec{B}}{\partial t} = \eta \nabla^2 \vec{B}$$ Diffusive Limit

14

Vorticity Analogy

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right) + \eta \nabla^2 \vec{B}$$

This equation is identical to the vorticity equation for barotropic fluids $\rho = \rho(P)$,

$$\frac{\partial \vec{\omega}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{\omega} \right) + \nu \nabla^2 \vec{\omega}$$

$$\vec{\omega} = \vec{\nabla} \times \vec{v}$$ Vorticity

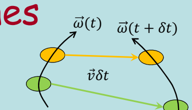
$$\nu = \frac{\mu}{\rho}$$ Kinematic viscosity

Therefore, the magnetic field evolves in the same fashion that vortex lines do.

15

Vortex Lines

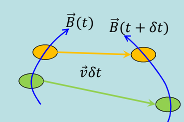
$$\frac{\partial \vec{\omega}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{\omega} \right) + \nu \nabla^2 \vec{\omega}$$



We know from fluid dynamics that if the fluid is inviscid, vortex lines follow the fluid.

- If a fluid parcel lies on a vortex line, it will forever stay on that line no matter how the parcel moves.
- The parcel and vortex line move together.
- If the fluid has viscosity, vortex lines and fluid parcel become decoupled and may slide through each other.

By analogy, we expect the magnetic field to move with the fluid as long as magnetic diffusion can be ignored. This property is called "flux freezing" in MHD. The field is said to be "frozen into the plasma".



16

Ideal MHD Limit

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right)$$

This equation describes a vector field that moves with the fluid.

This property was proved for the vorticity equation by Lord Kelvin with "Kelvin's Vorticity Theorem".

In MHD the same result was obtained by Hannes Alfvén and is called "Alfvén's Theorem of Flux Freezing".

17

Hannes Alfvén (1908-1995)

Alfvén was a Swedish electrical engineer and plasma physicist.

Aurorae
Van Allen Radiation Belts
Magnetic storms of Earth's magnetic field
Galactic plasma dynamics

Magnetohydrodynamics
(Nobel Prize 1970)

18

The Induction Equation

Magnetic Diffusion

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right) + \eta \nabla^2 \vec{B}$$

Flux Freezing
(the field moves with the fluid)

19

Alternate Form

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right) + \eta \nabla^2 \vec{B}$$

Since curls and cross products are annoying, often an alternate form of the induction equation is used. Use the following vector identity,

$$\vec{\nabla} \times \left(\vec{v} \times \vec{B} \right) = \vec{v} \left(\vec{\nabla} \cdot \vec{B} \right) - \vec{B} \left(\vec{\nabla} \cdot \vec{v} \right) + \left(\vec{B} \cdot \vec{\nabla} \right) \vec{v} - \left(\vec{v} \cdot \vec{\nabla} \right) \vec{B}$$

Solenoidal

Advection

$$\frac{D\vec{B}}{Dt} = -\vec{B} \left(\vec{\nabla} \cdot \vec{v} \right) + \left(\vec{B} \cdot \vec{\nabla} \right) \vec{v} + \eta \nabla^2 \vec{B}$$

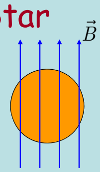
20

Example: Stellar Collapse to a Neutron Star

Consider a star with

$$R_* \approx 10^{11} \text{ cm}$$

$$B_* \approx 100 \text{ G}$$



$$\Phi = \pi R^2 B$$

If this star collapses to a neutron star (NS)

$$R_{NS} \approx 10^6 \text{ cm}$$

The neutron star's field strength can be estimated by noting that the magnetic flux is conserved (due to flux freezing)

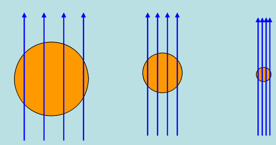
$$\Phi_{NS} = \Phi_*$$

$$\pi R_{NS}^2 B_{NS} = \pi R_*^2 B_* \rightarrow B_{NS} = \frac{R_*^2}{R_{NS}^2} B_* \approx 10^{12} \text{ G}$$

21

Flux Conservation

The same number of field lines pass through a smaller cross-sectional area



22