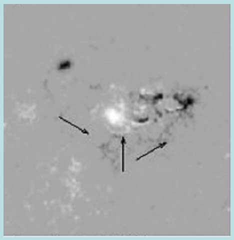
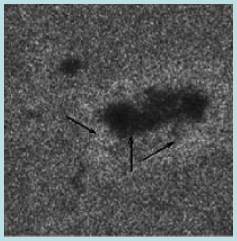
ASTR 7500: Solar & Stellar Magnetism

Hale CGEP Solar & Space Physics





Profs. Brad Hindman & Juri Toomre

Lecture 26 Thurs 25 Apr 2013

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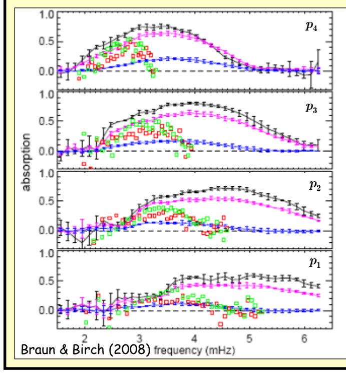
Sunspots Devour Acoustic Waves



Braun & Birch (2008)

Sunspots voracious absorb 50% or more of the acoustic energy that enters them.

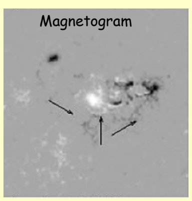
Hankel Decomposition

- Red – NOAA 5229
- Green – NOAA 5254

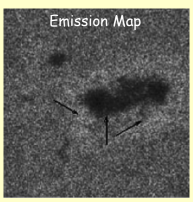
Holography

- Black - Umbrae
- Pink - Penumbrae
- Blue - Plage

Magnetogram

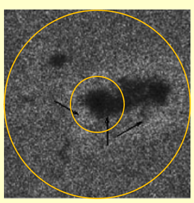


Emission Map



High-frequency (> 4 mHz) acoustic emission surrounds the activity. These are called "halos".

Fourier-Hankel Decomposition can pick up contribution from more than just the sunspot.



The annulus center can be contaminated by these halos.

Lecture 26
MHD Waves in Stratified Atmospheres

- Isothermal Atmosphere with a Uniform Magnetic Field
  - Characteristic Speeds
  - Linearization
- Coupled Wave Equations
- Alfvén Waves
- Magnetosonic Waves
  - Weak Field Limit
  - Strong Field Limit
- Mode Conversion

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Remember

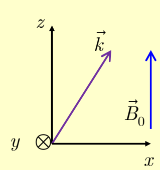
Alfvén Mode Synopsis

$\omega^2 = k_z^2 V_A^2$ Dispersion Relation

$\vec{v}_p \approx \frac{k_z}{k} V_A \hat{k}$ Phase Speed

$\vec{v}_g \approx V_A \hat{z}$ Group Speed

$\vec{v} = v\hat{y}$ Polarization



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Remember

Fast Mode Synopsis

$c_F^2 \equiv c_s^2 + V_A^2$

$\omega^2 = k^2 c_F^2 - k_z^2 c_T^2 + \ldots$ Asymptotic Dispersion Relation

$\vec{v}_p \approx c_F \hat{k}$ $\quad \vec{v}_g \approx c_F \hat{k}$ Phase and Group Speeds

Weak Field

$\frac{V_A^2}{c_s^2} \ll 1$

$\omega^2 = k^2 c_s^2 + \ldots$

$\vec{v} = v\hat{k} + \cdots$

~Acoustic Wave (Gas Pressure)

Strong Field

$\frac{V_A^2}{c_s^2} \gg 1$

$\omega^2 = k^2 V_A^2 + \ldots$

$\vec{v} = v\hat{x} + \cdots$

~Magnetic Wave (Magnetic Pressure and Tension)

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Remember

Slow Mode Synopsis

$$c_{\mathrm{T}}^2 \equiv \frac{c_s^2 V_A^2}{c_s^2 + V_A^2}$$

$$\omega^2 = k_z^2 c_{\mathrm{T}}^2 + \frac{k_z^4}{k^2}\frac{c_T^4}{c_F^2} + \cdots$$ Asymptotic Dispersion Relation

$$\vec{v}_{\mathrm{p}} \approx \frac{k_z}{k} c_{\mathrm{T}} \hat{k} \qquad \vec{v}_{\mathrm{g}} \approx c_{\mathrm{T}} \hat{z}$$ Phase and Group Speeds

Weak Field

$$\frac{V_A^2}{c_s^2} \ll 1 \qquad \omega^2 = k_z^2 V_A^2 + \cdots \qquad \vec{v} = v\,\hat{k}\times\hat{y} + \cdots$$

~Alfvénic

(Tension w/ counteracting pressures)

Strong Field

$$\frac{V_A^2}{c_s^2} \gg 1 \qquad \omega^2 = k_z^2 c_s^2 + \cdots \qquad \vec{v} = v\hat{z} + \cdots$$

~Ducted Acoustic (Gas Pressure)

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Isothermal Atmosphere with a Uniform Vertical Magnetic Field

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MHD Equations

We start from the MHD equations expressing the conservation of mass, energy, magnetic flux, and momentum. We ignore viscosity, thermal conduction, and other nonadiabatic processes.

Continuity Equation
$$\frac{D\rho}{Dt} = -\rho\vec{\nabla}\cdot\vec{v}$$ ← Fully Compressible

Energy Equation
$$\frac{DP}{Dt} = c_s^2\frac{D\rho}{Dt}$$ ← Adiabatic

Induction Equation
$$\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times\left(\vec{v}\times\vec{B}\right)$$ ← Ideal MHD

Momentum Equation
$$\rho\frac{D\vec{v}}{Dt} = -\vec{\nabla}P + \vec{g}\rho + \frac{(\vec{\nabla}\times\vec{B})\times\vec{B}}{4\pi}$$

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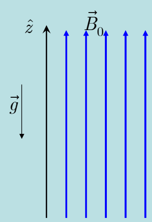
Isothermal Background

Let the background fluid be a plane-parallel isothermal atmosphere with constant gravity. Further, consider a constant background magnetic field of strength B_0, that points in the vertical z direction (antiparallel to gravity). Note that this magnetic field is force-free.



$$\vec{B}_0 = B_0\hat{z} = \text{constant}$$ Uniform field

$$P_0 = \hat{P}_0 e^{-z/H}$$
$$\rho_0 = \hat{\rho}_0 e^{-z/H}$$
Constant Density and Pressure Scale Heights
$$H^{-1} = \frac{\gamma g}{c_s^2}$$

$$T_0 = \text{constant}$$ Isothermal

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Acoustic-Gravity Waves

$$\left\{\frac{d^2}{dz^2} - \frac{1}{H}\frac{d}{dz} + \frac{1}{H^2}\frac{dH}{dz} + \frac{\omega^2}{c^2} + k_h^2\left(\frac{N^2}{\omega^2}-1\right)\right\}\left(c^2\chi\right) = 0$$

If the atmosphere is isothermal, the sound speed and the density scale heights are constants

$$\left[\frac{d^2}{dz^2} - \frac{1}{H}\frac{d}{dz} + \frac{\omega^2}{c^2} + k_h^2\left(\frac{N^2}{\omega^2}-1\right)\right]\left(\frac{dv_z}{dz} + ik_x v_x\right) = 0$$

In an isothermal atmosphere the same equation holds for each velocity component

$$\left[\frac{d^2}{dz^2} - \frac{1}{H}\frac{d}{dz} + \frac{\omega^2}{c^2} + k_h^2\left(\frac{N^2}{\omega^2}-1\right)\right]v_z = 0$$

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Characteristic Speeds

Since the atmosphere is isothermal, the sound speed is a constant. The Alfvén speed on the other hand varies with height in an exponential fashion.

$$c_s^2 = \text{constant}$$

$$V_A^2 = \frac{B_0^2}{4\pi\rho_0} \quad \xrightarrow{\rho_0 = \hat{\rho}_0 e^{-z/H}} \quad V_A^2 = \frac{B_0^2}{4\pi\hat{\rho}_0}e^{z/H} = \hat{V}_A^2\, e^{z/H}$$

The Alfvén speed is very large high in the atmosphere

The Alfvén speed is very small deep in the atmosphere

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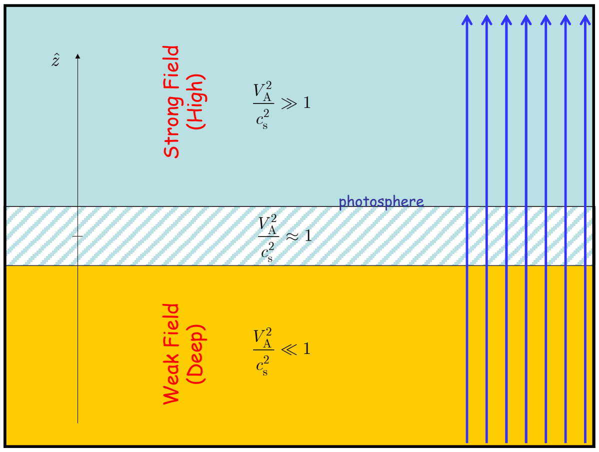

Consider Linear Waves

Linearize the MHD equations about the background atmosphere.

$$\vec{B}(\vec{x},t) = \vec{B}_0 + \vec{B}_1(\vec{x},t)$$

$$P(\vec{x},t) = P_0(z) + P_1(\vec{x},t)$$

$$\rho(\vec{x},t) = \rho_0(z) + \rho_1(\vec{x},t)$$

$$\vec{v}(\vec{x},t) = \vec{v}_1(\vec{x},t)$$

This subscript will be dropped from here on.

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Linearized MHD Equations

Since the atmosphere is horizontally homogeneous and the background magnetic field is constant, the linearized forms of the MHD equations are relatively simple.

Continuity Equation
$$\frac{\partial \rho_1}{\partial t} + v_z \frac{d\rho_0}{dz} = -\rho_0 \vec{\nabla} \cdot \vec{v}$$

Energy Equation
$$\frac{\partial P_1}{\partial t} + v_z \frac{dP_0}{dz} = -\rho_0 c_s^2 \vec{\nabla} \cdot \vec{v}$$

Induction Equation
$$\frac{\partial \vec{B}_1}{\partial t} = -\vec{B}_0 \left(\vec{\nabla} \cdot \vec{v}\right) + B_0 \frac{d\vec{v}}{dz}$$

Momentum Equation
$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla} P_1 - g\rho_1 \hat{z} + \frac{(\vec{\nabla} \times \vec{B}_1) \times \vec{B}_0}{4\pi}$$

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Plane Waves

Since the atmosphere is horizontally homogeneous and steady, the coefficients in the previous set of PDEs are functions of z alone. Thus, we should seek plane-wave solutions,

$$\rho_1(\vec{x},t) = \tilde{\rho}_1(z) \exp\left(ik_x x - i\omega t\right)$$

$$P_1(\vec{x},t) = \tilde{P}_1(z) \exp\left(ik_x x - i\omega t\right)$$

$$\vec{v}(\vec{x},t) = \tilde{\vec{v}}(z) \exp\left(ik_x x - i\omega t\right)$$

$$\vec{B}_1(\vec{x},t) = \tilde{\vec{B}}_1(z) \exp\left(ik_x x - i\omega t\right)$$

$\omega =$ Frequency

$k_x =$ Horizontal Wavenumber

Note, WLOG the horizontal wavenumber is only in the x-direction.

For simplicity, I will drop all of the tildes from here on forward.

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Fourier Transformed Equations

Insert the plane-wave functional form (or Fourier Transform the equations) to find the following

Continuity Equation
$$-i\omega\rho_1 = \frac{\rho_0}{H} v_z - \rho_0 \vec{\nabla} \cdot \vec{v}$$

Energy Equation
$$-i\omega P_1 = g\rho_0 v_z - \rho_0 c_s^2 \vec{\nabla} \cdot \vec{v}$$

Induction Equations
$$-i\omega B_{1x} = B_0 \frac{dv_x}{dz}$$ Horizontal x-component

$$-i\omega B_{1y} = B_0 \frac{dv_y}{dz}$$ Horizontal y-component

$$-i\omega B_{1z} = -ik_x B_0 v_x$$ Vertical component

Dilation
$$\vec{\nabla} \cdot \vec{v} = ik_x v_x + \frac{dv_z}{dz}$$ ← Note, no y velocity

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Linearized Momentum Equations

Momentum Equations

$$-i\omega\rho_0 v_x = -ik_x P_1 + \frac{B_0}{4\pi}\left(\frac{dB_{1x}}{dz} - ik_x B_{1z}\right)$$ Horizontal x-component

$$-i\omega\rho_0 v_y = \frac{B_0}{4\pi}\left(\frac{dB_{1y}}{dz}\right)$$ Horizontal y-component

$$-i\omega\rho_0 v_z = -\frac{dP_1}{dz} - g\rho_1$$ Vertical z-component

Magnetic Pressure (→ $-ik_x B_{1z}$)

Magnetic Tension (→ $\frac{dB_{1x}}{dz}$, $\frac{dB_{1y}}{dz}$)

Note that the vertical equation lacks any magnetic force terms.

Our ultimate goal is to use the continuity, energy and induction equations to eliminate all variables other than the velocities.

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Coupled Wave Equations

Goal

- Derive a set of coupled wave equations from the momentum equation
- Eliminate all variables other than the velocity components.
- Identify the MHD wave modes (as well as we can)

Vertical Momentum Equation

$$-i\omega\rho_0 v_z = -\frac{dP_1}{dz} - g\rho_1$$

The vertical momentum equation does not contain any magnetic terms. Furthermore, the energy and continuity equations are also devoid of magnetic terms.

Continuity

$$-i\omega\rho_1 = \frac{\rho_0}{H} v_z - \rho_0 \vec{\nabla}\cdot\vec{v}$$

Energy

$$-i\omega P_1 = g\rho_0 v_z - \rho_0 c_s^2 \vec{\nabla}\cdot\vec{v}$$

Therefore, we can play the same trick we used in Lecture 22: Acoustic-Gravity Waves to combine the pressure and gravity forces to reveal the buoyancy force

$$-\omega^2 v_z = i\omega \frac{d}{dz}\left(\frac{P_1}{\rho_0}\right) - \frac{c_s^2 N^2}{g}\vec{\nabla}\cdot\vec{v}$$

Same as slide 33 in Lecture 22: Acoustic-Gravity Waves

Vertical Equation continued

$$-\omega^2 v_z = i\omega \frac{d}{dz}\left(\frac{P_1}{\rho_0}\right) - \frac{c_s^2 N^2}{g}\vec{\nabla}\cdot\vec{v}$$ Vertical Momentum

To eliminate the pressure perturbation we use the energy equation.

$$-i\omega P_1 = g\rho_0 v_z - \rho_0 c_s^2 \vec{\nabla}\cdot\vec{v}$$ Energy

$$-i\omega \frac{P_1}{\rho_0} = g v_z - c_s^2 \vec{\nabla}\cdot\vec{v}$$

Combine this with the vertical momentum equation to find

$$-\omega^2 v_z = \frac{d}{dz}\left(c_s^2 \vec{\nabla}\cdot\vec{v} - g v_z\right) - \frac{c_s^2 N^2}{g}\vec{\nabla}\cdot\vec{v}$$

Vertical Momentum Equation

$$-\omega^2 v_z = \frac{d}{dz}\left(c_s^2 \vec{\nabla}\cdot\vec{v} - g v_z\right) - \frac{c_s^2 N^2}{g}\vec{\nabla}\cdot\vec{v}$$

Expand out the dilation in terms of the two velocity components, and collect terms involving v_x and v_z on different sides of the equation.

$$\vec{\nabla}\cdot\vec{v} = ik_x v_x + \frac{dv_z}{dz}$$

$$\left[c_s^2 \frac{d^2}{dz^2} - \left(\frac{c_s^2 N^2}{g} + g\right)\frac{d}{dz} + \omega^2\right] v_z = -ik_x c_s^2 \left(\frac{d}{dz} - \frac{N^2}{g}\right) v_x$$

Remember the definition for the buoyancy frequency $N^2 = g\left(\frac{1}{H} - \frac{g}{c_s^2}\right)$

$$\left[c_s^2 \frac{d^2}{dz^2} - \frac{c_s^2}{H}\frac{d}{dz} + \omega^2\right] v_z = -ik_x c_s^2 \left(\frac{d}{dz} - \frac{N^2}{g}\right) v_x$$

What does it all mean?

$$\left[c_s^2 \frac{d^2}{dz^2} - \frac{c_s^2}{H}\frac{d}{dz} + \omega^2\right] v_z = -ik_x c_s^2 \left(\frac{d}{dz} - \frac{N^2}{g}\right) v_x$$

Gas Pressure — Inertia — Buoyancy

x Horizontal Momentum Equation

$$-i\omega\rho_0 v_x = -ik_x P_1 + \frac{B_0}{4\pi}\left(\frac{dB_{1x}}{dz} - ik_x B_{1z}\right)$$

The x-horizontal momentum does possess magnetic terms. The dependence on the perturbed pressure and the perturbed magnetic field can be eliminated by using the energy and induction equations.

$$-i\omega B_{1x} = B_0 \frac{dv_x}{dz}$$ x Horizontal Induction

$$-i\omega B_{1z} = -ik_x B_0 v_x$$ Vertical Induction

$$-i\omega \frac{P_1}{\rho_0} = g v_z - c_s^2 \vec{\nabla}\cdot\vec{v}$$ Energy

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x Horizontal Momentum Equation

$$-i\omega \frac{P_1}{\rho_0} = g v_z - c_s^2 \vec{\nabla}\cdot\vec{v} \qquad -i\omega B_{1x} = B_0 \frac{dv_x}{dz} \qquad -i\omega B_{1z} = -ik_x B_0 v_x$$

$$-i\omega\rho_0 v_x = -ik_x P_1 + \frac{B_0}{4\pi}\left(\frac{dB_{1x}}{dz} - ik_x B_{1z}\right)$$

$$\left[V_A^2 \frac{d^2}{dz^2} - k_x^2\left(c_s^2 + V_A^2\right) + \omega^2\right] v_x = -ik_x\left(c_s^2\frac{d}{dz} - g\right) v_z$$

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What does it all mean?

$$\left[V_A^2 \frac{d^2}{dz^2} - k_x^2\left(c_s^2 + V_A^2\right) + \omega^2\right] v_x = -ik_x\left(c_s^2\frac{d}{dz} - g\right) v_z$$

Gas Pressure

Magnetic Tension

Magnetic Pressure

Inertia

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y Horizontal Momentum Equation

$$-i\omega\rho_0 v_y = \frac{B_0}{4\pi}\left(\frac{dB_{1y}}{dz}\right)$$

Since the horizontal wavevector only has an x component, the y momentum equation only contains a magnetic tension term.

$$-i\omega\rho_0 v_y = \frac{B_0}{4\pi}\left(\frac{dB_{1y}}{dz}\right) \qquad -i\omega B_{1y} = B_0\frac{dv_y}{dz}$$

y Induction equation

$$\left[V_A^2 \frac{d^2}{dz^2} + \omega^2\right] v_y = 0$$

Magnetic Tension

Inertia

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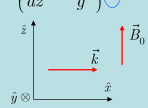
Three Momentum Equations

x $$\left[V_A^2 \frac{d^2}{dz^2} - k_x^2\left(c_s^2 + V_A^2\right) + \omega^2\right] v_x = -ik_x\left(c_s^2\frac{d}{dz} - g\right) v_z$$

y $$\left[V_A^2 \frac{d^2}{dz^2} + \omega^2\right] v_y = 0$$

z $$\left[c_s^2 \frac{d^2}{dz^2} - \frac{c_s^2}{H}\frac{d}{dz} + \omega^2\right] v_z = -ik_x c_s^2\left(\frac{d}{dz} - \frac{N^2}{g}\right) v_x$$



The y equation depends only on the y velocity v_y. Therefore, it decouples from the other two equations. The x and z equations are coupled with each other.

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Alfvén Waves

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Alfvén Waves

The y momentum equation decouples from the other two momentum equations. It has simple form and describes the shear Alfvén wave.

$$\left[V_A^2(z)\frac{d^2}{dz^2}+\omega^2\right]v_y=0$$

In our isothermal atmosphere, the Alfvén speed is an exponentially increasing function of height. So, the local wavelength becomes larger with height.

$$\frac{d^2 v_y}{dz^2}+\frac{\omega^2}{V_A^2(z)}v_y=0$$

$$\frac{d^2\psi}{dz^2}+K^2(z)v_y=0$$

- This wave has no turning points.
- It's group and phase velocity are parallel to the field.
- It's vertical propagation is independent of k_x.
- It's incompressive.

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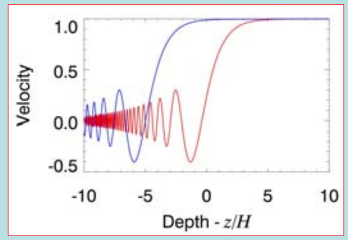
Alfvén Wave Wavefunction

$$\frac{d^2 v_y}{dz^2}+\frac{\omega^2}{V_A^2(z)}v_y=0$$

Since $V_A^2 \propto e^{z/H}$ this equation has an analytic solution.

$$v_y=A_1 J_0\left(2\Omega e^{-\frac{z}{2H}}\right)+A_2 Y_0\left(2\Omega e^{-\frac{z}{2H}}\right)$$

where $\Omega=\frac{\omega H}{c}$



$\Omega=1.0$

$\Omega=0.1$

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Magnetosonic Waves

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Magnetosonic Waves

The x and z momentum equations describe the propagation of the fast and slow magnetosonic waves.

$$\left[V_A^2\frac{d^2}{dz^2}-k_x^2\left(c_s^2+V_A^2\right)+\omega^2\right]v_x=-ik_x\left(c_s^2\frac{d}{dz}-g\right)v_z$$

$$\left[c_s^2\frac{d^2}{dz^2}-\frac{c_s^2}{H}\frac{d}{dz}+\omega^2\right]v_z=-ik_x c_s^2\left(\frac{d}{dz}-\frac{N^2}{g}\right)v_x$$

What can we say about these equations *without* actually solving them?

- The x and z equations are coupled.
  (The fast and slow magnetosonic modes are coupled.)
- The coefficients are nonconstant.
  (The solutions are not sinusoidal.)
- The set of equations is 4th order!
  (2 modes x 2 = 4th order)

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Weak and Strong Field Limits

$$\left[V_A^2\frac{d^2}{dz^2}-k_x^2\left(c_s^2+V_A^2\right)+\omega^2\right]v_x=-ik_x\left(c_s^2\frac{d}{dz}-g\right)v_z$$

$$\left[c_s^2\frac{d^2}{dz^2}-\frac{c_s^2}{H}\frac{d}{dz}+\omega^2\right]v_z=-ik_x c_s^2\left(\frac{d}{dz}-\frac{N^2}{g}\right)v_x$$

In general, these equations do not have a simple solution because the fast and slow modes are coupled (and the equation is 4th order). However, when the Alfvén speed is either very small or very large compared to the sound speed, the two modes decouple!

<u>Deep in the atmosphere</u>

$z\rightarrow-\infty$

$\frac{V_A^2}{c_s^2}\ll 1$ Weak Field Limit

<u>High in the atmosphere</u>

$z\rightarrow\infty$

$\frac{V_A^2}{c_s^2}\gg 1$ Strong Field Limit

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Magnetosonic Waves

Weak Field Limit
(Deep)

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$\frac{V_A^2}{c_s^2} \ll 1$

Weak Field Limit – Fast Mode

$$\left[V_A^2 \frac{d^2}{dz^2} - k_x^2\left(c_s^2 + V_A^2\right) + \omega^2\right] v_x = -ik_x\left(c_s^2 \frac{d}{dz} - g\right) v_z$$

$$\left[c_s^2 \frac{d^2}{dz^2} - \frac{c_s^2}{H}\frac{d}{dz} + \omega^2\right] v_z = -ik_x c_s^2\left(\frac{d}{dz} - \frac{N^2}{g}\right) v_x$$

When the Alfvén speed is small, we can recover the fast mode by simply neglecting all terms with the Alfvén speed. The x momentum equation simplifies (but the z does not).

$$\left[V_A^2 \frac{d^2}{dz^2} - k_x^2\left(c_s^2 + V_A^2\right) + \omega^2\right] v_x = -ik_x\left(c_s^2 \frac{d}{dz} - g\right) v_z$$

$$\left(\omega^2 - k_x^2 c_s^2\right) v_x = -ik_x\left(c_s^2 \frac{d}{dz} - g\right) v_z$$

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$\frac{V_A^2}{c_s^2} \ll 1$

Deep – Fast Mode

$$\left(\omega^2 - k_x^2 c_s^2\right) v_x = -ik_x\left(c_s^2 \frac{d}{dz} - g\right) v_z$$

$$\left[c_s^2 \frac{d^2}{dz^2} - \frac{c_s^2}{H}\frac{d}{dz} + \omega^2\right] v_z = -ik_x c_s^2\left(\frac{d}{dz} - \frac{N^2}{g}\right) v_x$$

These two equation can be combined together (by eliminating v_x) to obtain a readily recognized equation.

$$\left[\frac{d^2}{dz^2} - \frac{1}{H}\frac{d}{dz} + \frac{\omega^2}{c_s^2} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right)\right] v_z = 0$$

This is the equation for acoustic-gravity waves in an isothermal atmosphere in the absence of a magnetic field! Note, this is a second order ODE.

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$\frac{V_A^2}{c_s^2} \ll 1$

Weak Field Limit – Slow Mode

$$\left[V_A^2 \frac{d^2}{dz^2} - k_x^2\left(c_s^2 + V_A^2\right) + \omega^2\right] v_x = -ik_x\left(c_s^2 \frac{d}{dz} - g\right) v_z$$

$$\left[c_s^2 \frac{d^2}{dz^2} - \frac{c_s^2}{H}\frac{d}{dz} + \omega^2\right] v_z = -ik_x c_s^2\left(\frac{d}{dz} - \frac{N^2}{g}\right) v_x$$

When the Alfvén speed is small, recovering the slow mode is trickier. The slow mode is obtained by recognizing that it is short wavelength. Thus, we treat d/dz as a term of order $1/V_A$. (In the absence of gravity the waves would be Alfvénic.)

$$\left[V_A^2 \frac{d^2}{dz^2} - k_x^2\left(c_s^2 + V_A^2\right) + \omega^2\right] v_x = -ik_x\left(c_s^2 \frac{d}{dz} - g\right) v_z$$

$$\left[c_s^2 \frac{d^2}{dz^2} - \frac{c_s^2}{H}\frac{d}{dz} + \omega^2\right] v_z = -ik_x c_s^2\left(\frac{d}{dz} - \frac{N^2}{g}\right) v_x$$

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$\frac{V_A^2}{c_s^2} \ll 1$

Deep – Slow Mode

$$\left[V_A^2 \frac{d^2}{dz^2} - k_x^2 c_s^2 + \omega^2\right] v_x = -ik_x c_s^2 \frac{d}{dz} v_z$$

$$c_s^2 \frac{d^2}{dz^2} v_z = -ik_x c_s^2 \frac{d}{dz} v_x$$

Examine this last equation in a bit more detail. If we divide by the square of the sound speed and integrate in depth once, we discover that the dilation is zero.

$$c_s^2 \frac{d^2}{dz^2} v_z = -ik_x c_s^2 \frac{d}{dz} v_x \quad \Rightarrow \quad \frac{d^2 v_z}{dz^2} + ik_x \frac{dv_x}{dz} = 0$$

$$ik_x v_x + \frac{dv_z}{dz} = \vec{\nabla}\cdot\vec{v} = 0$$

The slow mode deep in the atmosphere is incompressive!

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$\frac{V_A^2}{c_s^2} \ll 1$

Deep – Slow Mode

$$\left[V_A^2 \frac{d^2}{dz^2} - k_x^2 c_s^2 + \omega^2\right] v_x = -ik_x c_s^2 \frac{d}{dz} v_z$$

$$ik_x v_x + \frac{dv_z}{dz} = \vec{\nabla}\cdot\vec{v} = 0$$

We can now use the incompressibility condition to eliminate v_z from the x momentum equation.

$$\left[V_A^2(z) \frac{d^2}{dz^2} + \omega^2\right] v_x = 0$$

Huh? The slow magnetosonic wave looks like a Alfvén wave in this limit. Except that it has (tiny) motion along the magnetic field!

$$\frac{dv_z}{dz} = -ik_x v_x$$

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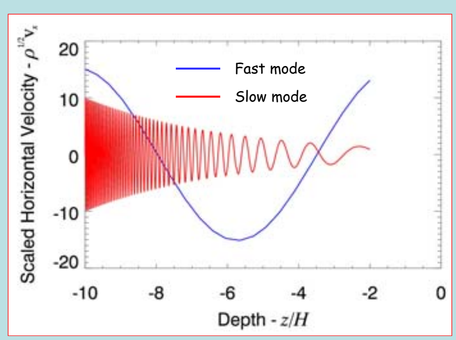
Weak Field - Wavefunctions



$$\frac{\omega H}{c} = 1.0$$

$$k_x H = 0.1$$

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$\hat{z}$

Strong Field

$\frac{V_A^2}{c_s^2} \gg 1$

$\frac{V_A^2}{c_s^2} \approx 1$

Weak Field

$\frac{V_A^2}{c_s^2} \ll 1$

Fast Waves

- Acoustic-Gravity Wave
- Propagates across and along field lines (doesn't care about the field)

$$k_z^2(z) = \frac{\omega^2 - \omega_c^2}{c_s^2} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right)$$

Slow Waves

- Alfvénic Wave (incompressive)
- Propagates at the Alfvén speed
- Propagates along field lines

$$k_z^2(z) = \frac{\omega^2}{V_A^2}$$

Magnetosonic Waves

Strong Field Limit (High)

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Strong Field Limit – Fast Mode

$\frac{V_A^2}{c_s^2} \gg 1$

$$\left[V_A^2 \frac{d^2}{dz^2} - k_x^2\left(c_s^2 + V_A^2\right) + \omega^2\right] v_x = -ik_x\left(c_s^2 \frac{d}{dz} - g\right) v_z$$

$$\left[c_s^2 \frac{d^2}{dz^2} - \frac{c_s^2}{H}\frac{d}{dz} + \omega^2\right] v_z = -ik_x c_s^2\left(\frac{d}{dz} - \frac{N^2}{g}\right) v_x$$

When the Alfvén speed is large, we can recover the fast mode by neglecting all gas pressure and buoyancy forces in the first equation.

$$\left[V_A^2 \frac{d^2}{dz^2} - k_x^2\left(c_s^2 + V_A^2\right) + \omega^2\right] v_x = -ik_x\left(c_s^2 \frac{d}{dz} - g\right) v_z$$

$$\left[\frac{d^2}{dz^2} + \frac{\omega^2}{V_A^2} - k_x^2\right] v_x = 0$$

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High – Fast Mode

$\frac{V_A^2}{c_s^2} \gg 1$

$$\left[\frac{d^2}{dz^2} + \frac{\omega^2}{V_A^2} - k_x^2\right] v_x = 0$$

The fast wave in this limit is purely a magnetic wave. It possesses the following properties:

- There is a turning point (a refraction point). $k_z^2(z) = \frac{\omega^2}{V_A^2(z)} - k_x^2$
- The wave propagates at the Alfvén speed.
- The propagation is NOT parallel to the magnetic field. The equation depends on the transverse wavenumber.
- The velocity is composed of both horizontal and vertical motions.

$$\left[c_s^2 \frac{d^2}{dz^2} - \frac{c_s^2}{H}\frac{d}{dz} + \omega^2\right] v_z = -ik_x c_s^2\left(\frac{d}{dz} - \frac{N^2}{g}\right) v_x$$

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Strong Field Limit – Slow Mode

$\frac{V_A^2}{c_s^2} \gg 1$

$$\left[V_A^2 \frac{d^2}{dz^2} - k_x^2\left(c_s^2 + V_A^2\right) + \omega^2\right] v_x = -ik_x\left(c_s^2 \frac{d}{dz} - g\right) v_z$$

$$\left[c_s^2 \frac{d^2}{dz^2} - \frac{c_s^2}{H}\frac{d}{dz} + \omega^2\right] v_z = -ik_x c_s^2\left(\frac{d}{dz} - \frac{N^2}{g}\right) v_x$$

When the Alfvén speed is large, we can recover the slow mode by assuming that the motion is transverse to the field, i.e., v_x is much smaller than v_z.

$$\left[c_s^2 \frac{d^2}{dz^2} - \frac{c_s^2}{H}\frac{d}{dz} + \omega^2\right] v_z = -ik_x c_s^2\left(\frac{d}{dz} - \frac{N^2}{g}\right) v_x$$

$$\left[\frac{d^2}{dz^2} - \frac{1}{H}\frac{d}{dz} + \frac{\omega^2}{c_s^2}\right] v_z = 0$$

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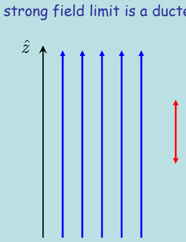
High – Slow Mode

$\frac{V_A^2}{c_s^2} \gg 1$

$$\left[\frac{d^2}{dz^2} - \frac{1}{H}\frac{d}{dz} + \frac{\omega^2}{c_s^2}\right] v_z = 0$$

This is the same equation one would get for acoustic-gravity waves if we only considered vertically propagating waves $k_h = 0$. So, the slow mode in the strong field limit is a ducted acoustic wave.

$$k_z^2(z) = \frac{\omega^2 - \omega_c^2}{c_s^2}$$



Acoustic wave motions are forced to align with the field lines since the field is so strong.

Fluid parcels on different field lines are disconnected from each other. (The dispersion relation doesn't depend on k_x).

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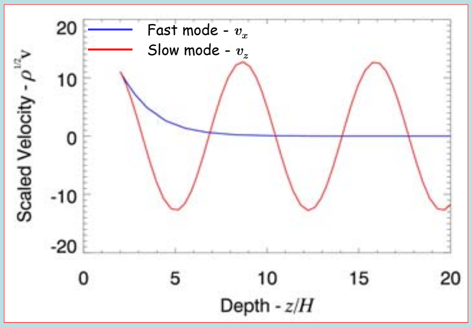
Strong Field - Wavefunctions



$\frac{\omega H}{c} = 1.0$

$k_x H = 0.1$

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$\hat{z}$

Strong Field $\frac{V_A^2}{c_s^2} \gg 1$

Fast Waves

- Magnetic Wave
- Propagates at the Alfvén speed
- Propagates across and along field lines

$k_z^2(z) = \frac{\omega^2}{V_A^2} - k_x^2$

Slow Waves

- Ducted acoustic wave
- Propagates at the sound speed
- Propagates along field lines

$k_z^2(z) = \frac{\omega^2 - \omega_c^2}{c_s^2}$

$\frac{V_A^2}{c_s^2} \approx 1$ What about Here?

Weak Field $\frac{V_A^2}{c_s^2} \ll 1$

Fast Waves

- Acoustic-Gravity Wave
- Propagates across and along field lines (doesn't care about the field)

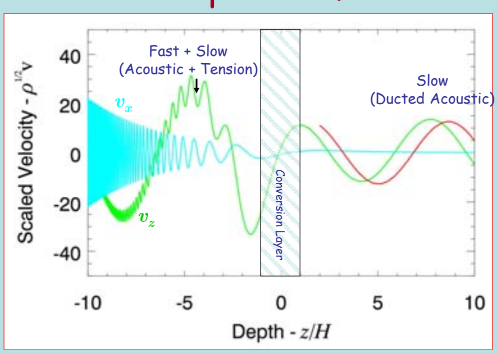
$k_z^2(z) = \frac{\omega^2 - \omega_c^2}{c_s^2} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right)$

Slow Waves

- Alfvénic Wave
- Propagates at the Alfvén speed
- Propagates along field lines

$k_z^2(z) = \frac{\omega^2}{V_A^2}$

Mixed-up Wavefunction



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Mode Conversion

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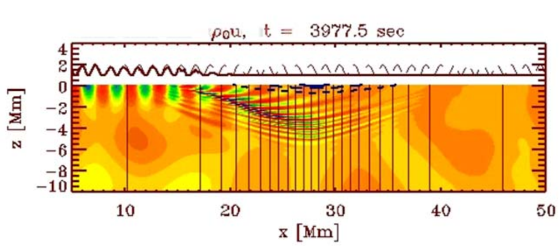

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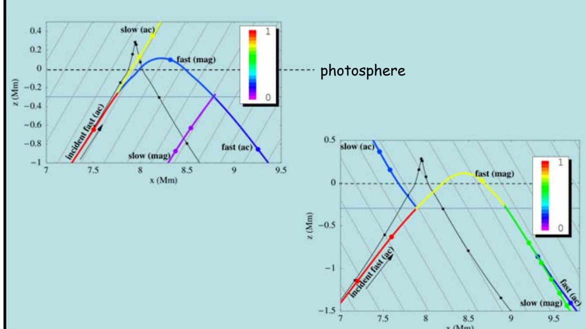
Ray splitting



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