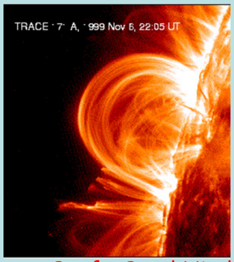
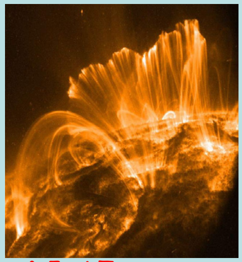
ASTR 7500: Solar & Stellar Magnetism

Hale CGEP Solar & Space Physics





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Lecture 25 Tues 23 Apr 2013

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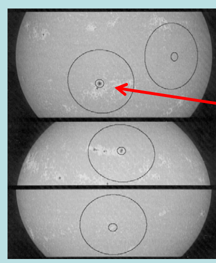
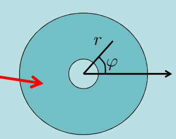
Lecture 25
MHD Waves

- Scattering and Absorption of Waves by Sunspots
- MHD Waves in a Homogeneous Atmosphere
  - Linearization
  - Alfvén Speed
  - Deriving a matrix equation
- MHD Wave Modes
  - Three wave modes (Alfvén, Fast Magnetosonic, Slow Magnetosonic)
  - Dispersion Relations
  - Polarization
- Shear Alfvén Wave
  - Properties of the Shear Alfvén Wave
- Fast and Slow Magnetosonic Waves
  - Pressure and magnetic forces
  - Why is the fast mode fast and the slow mode slow?

Scattering and Absorption of Acoustic Waves by Sunspots

Scattering Measurements in Cylindrical Coordinates





Transform the observed Dopplergrams into polar coordinates centered on sunspot (and in quiet sun as a control).

$$v_{los}(\vec{x},t) = v_{los}(r,\varphi,t)$$

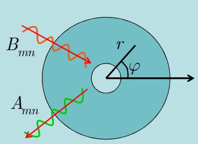
Fourier-Hankel Decomposition

Decompose into cylindrical waves

$$v_{los}(r,\varphi,t) = \int \sum_m \sum_n \left[A_{mn}(\omega) H_m^{(1)}(k_n r) + B_{mn}(\omega) H_m^{(2)}(k_n r) \right] e^{im\varphi} e^{-i\omega t}\, d\omega$$

Azimuthal Order — Radial Order — Outward Waves — Inward Waves

A_m and B_m are the amplitudes of the outward and inward propagating waves, respectively.



Hankel Functions

$$H_m^{(1,2)}(x) \equiv J_m(x) \pm i Y_m(x)$$

Absorption

$$v_{los}(r,\varphi,t) = \int \sum_m \sum_n \left[A_{mn}(\omega) H_m^{(1)}(k_n r) + B_m(\omega) H_m^{(2)}(k_l r) \right] e^{im\varphi} e^{-i\omega t}\, d\omega$$

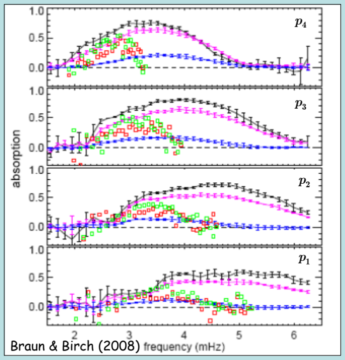
If $\left|A_{mn}(\omega)\right| \neq \left|B_{mn}(\omega)\right|$ the sunspot is either:

- redistributing the energy between modes (i.e., inward p_1 is scattered into outward p_2 or f)
- Destroying the acoustic wave energy

These processes are generically called absorption and can be observationally characterized by an *absorption coefficient*

$$\alpha_{mn}(\omega) \equiv \frac{\left|A_{mn}(\omega)\right|^2 - \left|B_{mn}(\omega)\right|^2}{\left|A_{mn}(\omega)\right|^2}$$

Sunspots Devour Acoustic Waves



Braun & Birch (2008)

Sunspots voracious absorb 50% or more of the acoustic energy that enters them.

Hankel Decomposition

Red – NOAA 5229

Green – NOAA 5254

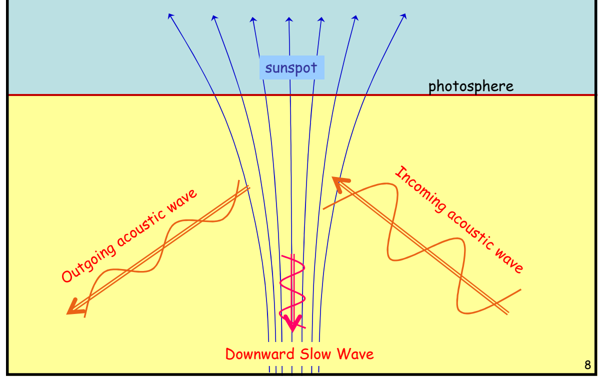
Holography

Black - Umbrae

Pink - Penumbrae

Blue - Plage

Preview – Mode Conversion



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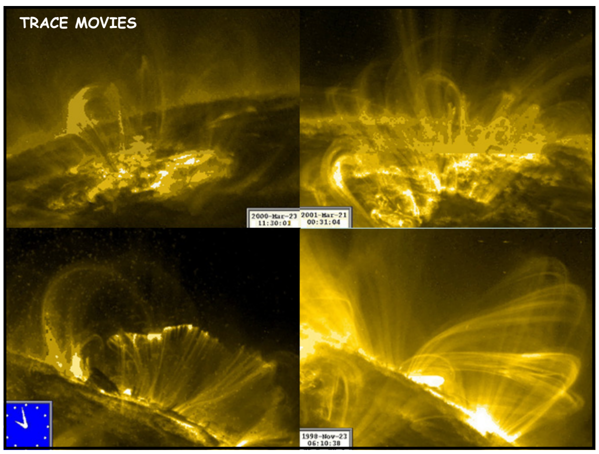

MHD Waves in a Homogeneous Atmosphere

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MHD Waves

We start from the MHD equations expressing the conservation of mass, momentum and energy. We ignore viscosity, gravity, thermal conduction and other nonadiabatic processes.

Continuity Equation
$$\frac{D\rho}{Dt} = -\rho\vec{\nabla}\cdot\vec{v}$$
← Fully Compressible

Momentum Equation
$$\rho\frac{D\vec{v}}{Dt} = -\vec{\nabla}P + \frac{(\vec{\nabla}\times\vec{B})\times\vec{B}}{4\pi}$$

$$c_s^2 = \frac{\gamma P}{\rho}$$

Energy Equation
$$\frac{DP}{Dt} = c_s^2\frac{D\rho}{Dt}$$
← Adiabatic

Induction Equation
$$\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times\left(\vec{v}\times\vec{B}\right)$$
← Ideal MHD

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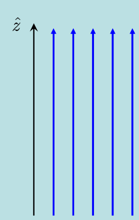
Linearize About a Homogeneous Background

Let the background fluid be stationary and homogeneous, with constant density ρ_0 and pressure P_0 as a function of position. Further, consider a constant background magnetic field of strength B_0, that points in the z direction.



$$\vec{B} = B_0\hat{z} = \text{constant}$$
$$P_0 = \text{constant}$$
$$\rho_0 = \text{constant}$$

$$\vec{B}(\vec{x},t) = \vec{B}_0 + \vec{B}_1(\vec{x},t)$$
$$P(\vec{x},t) = P_0 + P_1(\vec{x},t)$$
$$\rho(\vec{x},t) = \rho_0 + \rho_1(\vec{x},t)$$
$$\vec{v}(\vec{x},t) = \vec{v}_1(\vec{x},t)$$

Background Media is Homogeneous

This subscript will be dropped from here on.

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Linearized MHD Equations

Since the atmosphere is homogeneous (without gravitational stratification) and the background magnetic field is constant, the linearized form of the MHD equations is relatively simple.

Continuity Equation
$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \vec{\nabla} \cdot \vec{v}$$

Momentum Equation
$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla} P_1 + \frac{(\vec{\nabla} \times \vec{B}_1) \times \vec{B}_0}{4\pi}$$

Energy Equation
$$\frac{\partial P_1}{\partial t} = c_s^2 \frac{\partial \rho_1}{\partial t}$$

Induction Equation
$$\frac{\partial \vec{B}_1}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B}_0\right)$$

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Plane Waves

Since the atmosphere is homogeneous, all of the coefficients in the previous set of PDEs are constants. Thus, we should seek plane-wave solutions,

$$\rho_1(\vec{x},t) = \tilde{\rho}_1 \exp\left(i\vec{k} \cdot \vec{x} - i\omega t\right)$$
$$P_1(\vec{x},t) = \tilde{P}_1 \exp\left(i\vec{k} \cdot \vec{x} - i\omega t\right)$$
$$\vec{v}(\vec{x},t) = \tilde{\vec{v}} \exp\left(i\vec{k} \cdot \vec{x} - i\omega t\right)$$
$$\vec{B}_1(\vec{x},t) = \tilde{\vec{B}}_1 \exp\left(i\vec{k} \cdot \vec{x} - i\omega t\right)$$

$\omega =$ Frequency

$\vec{k} =$ Wavenumber

For simplicity, I will drop all of the tildes from here on forward.

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Fourier Transformed Equations

Insert the plane-wave functional form (or Fourier Transform the equations) to find the following

Continuity
$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \vec{\nabla} \cdot \vec{v} \quad \rightarrow \quad -i\omega \rho_1 = -i\rho_0 \vec{k} \cdot \vec{v}$$

Momentum
$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla} P_1 + \frac{(\vec{\nabla} \times \vec{B}_1) \times \vec{B}_0}{4\pi} \quad \rightarrow \quad -i\omega \rho_0 \vec{v} = -i\vec{k} P_1 + \frac{(i\vec{k} \times \vec{B}_1) \times \vec{B}_0}{4\pi}$$

Energy
$$\frac{\partial P_1}{\partial t} = c_s^2 \frac{\partial \rho_1}{\partial t} \quad \rightarrow \quad -i\omega P_1 = -i\omega c_s^2 \rho_1$$

Induction
$$\frac{\partial \vec{B}_1}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B}_0\right) \quad \rightarrow \quad -i\omega \vec{B}_1 = i\vec{k} \times \left(\vec{v} \times \vec{B}_0\right)$$

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Reduce to a Single Equation

Our goal is to eliminate every variable except the *velocity*.

We can eliminate the pressure perturbation in favor of the density perturbation through the energy equation

$$-i\omega P_1 = -i\omega c_s^2 \rho_1 \quad \rightarrow \quad P_1 = c_s^2 \rho_1$$

We can eliminate the density perturbation through the use of the continuity equation

$$-i\omega \rho_1 = -i\rho_0 \vec{k} \cdot \vec{v} \quad \rightarrow \quad \rho_1 = \frac{\rho_0}{\omega} \vec{k} \cdot \vec{v}$$

The induction equation can be used to eliminate the perturbed magnetic field

$$-i\omega \vec{B}_1 = i\vec{k} \times \left(\vec{v} \times \vec{B}_0\right) \quad \rightarrow \quad \vec{B}_1 = -\frac{\vec{k}}{\omega} \times \left(\vec{v} \times \vec{B}_0\right)$$

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Alfvén Velocity

$$P_1 = c_s^2 \rho_1$$
$$\rho_1 = \frac{\rho_0}{\omega} \vec{k} \cdot \vec{v}$$

$$\vec{B}_1 = -\frac{\vec{k}}{\omega} \times \left(\vec{v} \times \vec{B}_0\right)$$

$$-i\omega \rho_0 \vec{v} = -i\vec{k} P_1 + \frac{(i\vec{k} \times \vec{B}_1) \times \vec{B}_0}{4\pi}$$

Alfvén Velocity
$$\vec{V}_A = \frac{\vec{B}_0}{\sqrt{4\pi \rho_0}}$$

$$\left[\omega^2 - (\vec{k} \cdot \vec{V}_A)^2\right]\vec{v} = \left[\left(c_s^2 + V_A^2\right)(\vec{k} \cdot \vec{v}) - (\vec{k} \cdot \vec{V}_A)(\vec{V}_A \cdot \vec{v})\right]\vec{k} - (\vec{k} \cdot \vec{V}_A)(\vec{k} \cdot \vec{v})\vec{V}_A$$

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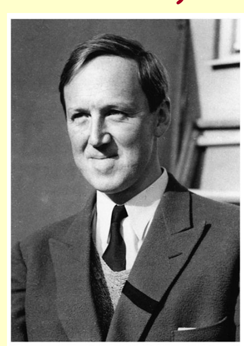
Hannes Alfvén (1908-1995)

Alfvén was a Swedish electrical engineer and plasma physicist.

- Aurorae
- Van Allen Radiation Belts
- Magnetic storms of Earth's magnetic field
- Galactic plasma dynamics

- Magnetohydrodynamics
  - (Nobel Prize 1970)



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Simplify

$$\left[\omega^2 - (\vec{k}\cdot\vec{V}_A)^2\right]\vec{v} = \left[\left(c_s^2 + V_A^2\right)(\vec{k}\cdot\vec{v}) - (\vec{k}\cdot\vec{V}_A)(\vec{V}_A\cdot\vec{v})\right]\vec{k} - (\vec{k}\cdot\vec{V}_A)(\vec{k}\cdot\vec{v})\vec{V}_A$$

Remember that the background magnetic field points in the z direction.

$$\left.\begin{aligned}\vec{B}_0 &= B_0\hat{z}\\ \vec{V}_A &= V_A\hat{z}\end{aligned}\right\} \longrightarrow \left\{\begin{aligned}\vec{k}\cdot\vec{V}_A &= k_z V_A\\ \vec{V}_A\cdot\vec{v} &= V_A v_z\end{aligned}\right.$$

$$\boxed{\left(\omega^2 - k_z^2V_A^2\right)\vec{v} = \left[\left(c_s^2 + V_A^2\right)(\vec{k}\cdot\vec{v}) - k_z V_A^2 v_z\right]\vec{k} - k_z V_A^2(\vec{k}\cdot\vec{v})\hat{z}}$$

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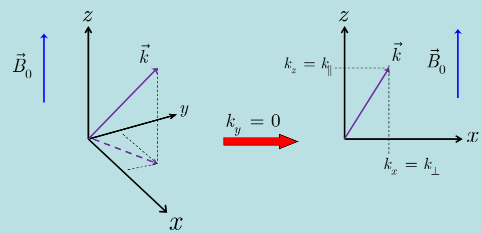
Horizontal Isotropy

We can further simplify be noting that x and y are interchangeable. Therefore, without loss of generality we may assume $k_y = 0$.



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Matrix Formulation

$$\left(\omega^2 - k_z^2V_A^2\right)\vec{v} = \left[\left(c_s^2 + V_A^2\right)(\vec{k}\cdot\vec{v}) - k_z V_A^2 v_z\right]\vec{k} - k_z V_A^2(\vec{k}\cdot\vec{v})\hat{z}$$

This equation is actually three separate equations, one for each component. Those three equations are coupled and can be written in a matrix form.

$$\mathbb{A}\vec{v} = -\omega^2\vec{v}$$

$$\mathbb{A} = \begin{bmatrix} -k_z^2V_A^2 - k_x^2\left(c_s^2 + V_A^2\right) & 0 & -k_x k_z c_s^2 \\ 0 & -k_z^2V_A^2 & 0 \\ -k_x k_z c_s^2 & 0 & -k_z^2 c_s^2 \end{bmatrix}$$

Zeros because $k_y = 0$

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MHD Wave Modes

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Eigenproblem

$$\mathbb{A}\vec{v} = -\omega^2\vec{v}$$

This is an eigenvalue-eigenvector problem.

- Since the matrix is 3x3, there are three eigenvalues and three eigenvectors. Each corresponds to a separate wave mode.
- The three eigenvalues ω^2 provide the dispersion relations.
- The eigenvectors provide the polarizations.
- The eigenvectors are orthogonal, and any disturbance can be expressed as a linear combination of the three wave modes.

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Dispersion Relations - Eigenvalues

$$\mathbb{A}\vec{v} = -\omega^2\vec{v} \longrightarrow \left(\mathbb{A} + \omega^2\mathbb{I}\right)\vec{v} = 0$$

If this matrix equation is to have a solution, the determinant of the matrix must vanish.

$$\det\left(\mathbb{A} + \omega^2\mathbb{I}\right) = 0$$

After only marginal algebra we obtain the dispersion relation

$$\boxed{\left(\omega^2 - k_z^2V_A^2\right)\left[\omega^4 - k^2\left(c_s^2 + V_A^2\right)\omega^2 + k^2k_z^2c_s^2V_A^2\right] = 0}$$

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Three Wave Modes

$$\left(\omega^2 - k_z^2 V_A^2\right)\left[\omega^4 - k^2\left(c_s^2 + V_A^2\right)\omega^2 + k^2 k_z^2 c_s^2 V_A^2\right] = 0$$

This equation is cubic in ω^2. Thus, there are three unique solutions for ω^2, and correspondingly three unique wave modes.

One solution satisfies

$$\omega^2 - k_z^2 V_A^2 = 0$$

Shear Alfvén Wave

Two solutions satisfy

$$\omega^4 - k^2\left(c_s^2 + V_A^2\right)\omega^2 + k^2 k_z^2 c_s^2 V_A^2 = 0$$

Fast and Slow Magnetosonic Waves

$$\omega^2 = \frac{k^2}{2}\left(c_s^2 + V_A^2\right) \pm \frac{k^2}{2}\sqrt{\left(c_s^2 + V_A^2\right)^2 - 4\frac{k_z^2}{k^2} c_s^2 V_A^2}$$

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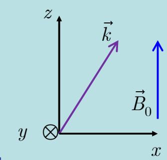
Polarizations - Eigenvectors

$$\left(\mathbb{A} + \omega^2 \mathbb{I}\right)\vec{v} = \begin{bmatrix} \omega^2 - k_z^2 V_A^2 - k_x^2\left(c_s^2 + V_A^2\right) & 0 & -k_x k_z c_s^2 \\ 0 & \omega^2 - k_z^2 V_A^2 & 0 \\ -k_x k_z c_s^2 & 0 & \omega^2 - k_z^2 c_s^2 \end{bmatrix}\vec{v} = 0$$

The three eigenvectors give the solution for the velocity for each wave mode.



The Shear Alfvén wave is polarized in the y direction

$$\vec{v} = \hat{y}$$

The magnetosonic waves have polarization in the x-z plane.

$$\vec{v} = \left(\omega_{f,s}^2 - k_z^2 c_s^2\right)\hat{x} + k_x k_z c_s^2\,\hat{z}$$

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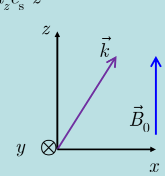
Orthogonality of Wave Polarizations

$$\mathbb{A}\vec{v} = -\omega^2 \vec{v} \qquad \vec{v}_{f,s} = \left(\omega_{f,s}^2 - k_z^2 c_s^2\right)\hat{x} + k_x k_z c_s^2\,\hat{z}$$

$$\vec{v}_A = \hat{y}$$



The Alfvén wave and the magnetosonic waves have velocities that are all mutually orthogonal. This is a consequence of the velocities being eigenvectors.

$$\left(\vec{v}_A \cdot \vec{v}_f\right) = \left(\vec{v}_A \cdot \vec{v}_s\right) = \left(\vec{v}_f \cdot \vec{v}_s\right) = 0$$

Since the three polarizations are mutually orthogonal, one can construct any disturbance as a linear superposition of Alfvén waves, fast magnetosonic waves, and slow magnetosonic waves.

$$\vec{v} = A_A \vec{v}_A + A_f \vec{v}_f + A_s \vec{v}_s$$

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Shear Alfvén Wave

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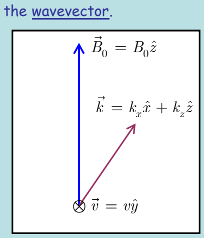
Shear Alfvén Wave

The Shear Alfvén wave satisfies the dispersion relation.

$$\omega^2 - k_z^2 V_A^2 = 0$$

The polarization of the eigenvector is purely in the y direction, perpendicular to both the magnetic field and the wavevector.

$$\vec{v} = U_A \hat{y}$$



The wave is incompressive.

$$\vec{\nabla} \cdot \vec{v} = i\vec{k} \cdot \vec{v} = k_y v_y = 0$$

$$\rho_1 = 0$$

$$P_1 = 0$$

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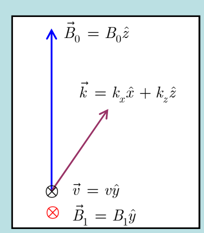
Alfvén Waves are Transverse

The perturbed magnetic field is also purely in the y direction. This can be shown using the induction equation.

$$\vec{B}_1 = -\frac{\vec{k}}{\omega} \times \left(\vec{v} \times \vec{B}_0\right)$$

$\vec{v} = U_A \hat{y}$

$$\vec{B}_1(\vec{x}, t) = -\frac{k_z B_0}{\omega} U_A\ \hat{y}$$



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Alfvén Waves are Tension Waves

Since Alfvén waves are incompressive, they lack perturbations to the magnetic pressure and the gas pressure. Thus, the restoring force must be magnetic tension.

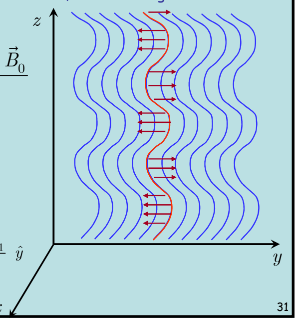
$$-i\omega\rho_0\vec{v} = -i\vec{k}P_1 + \frac{(i\vec{k}\times\vec{B}_1)\times\vec{B}_0}{4\pi}$$

Magnetic Pressure

$$-\vec{\nabla}\frac{B^2}{8\pi} = -\vec{k}\frac{\vec{B}_0\cdot\vec{B}_1}{4\pi} = 0$$

The tension force

$$\frac{\left(\vec{B}\cdot\vec{\nabla}\right)\vec{B}}{4\pi} = \frac{\left(\vec{B}_0\cdot i\vec{k}\right)\vec{B}_1}{4\pi} = ik_z\frac{B_0B_1}{4\pi}\hat{y}$$



Alfvén Waves Travel along Field Lines

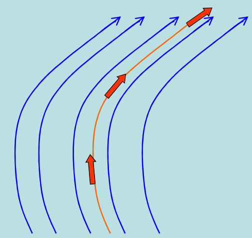
$$\omega^2 - k_z^2V_A^2 = 0$$

The dispersion relation is only a function of k_z. Thus, the group velocity is parallel to the magnetic field.

$$\omega = k_zV_A$$

$$\vec{v}_{\text{group}} = \vec{\nabla}_k\omega = V_A\hat{z}$$

Thus energy flows along the field lines and the waves propagate along the field. This property even holds in background configurations with curved field lines.



Magnetosonic Waves (or Magnetoacoustic)

Magnetosonic Waves

The two magnetosonic waves satisfy the dispersion relation

$$\omega^4 - k^2\left(c_s^2 + V_A^2\right)\omega^2 + k^2k_z^2c_s^2V_A^2 = 0$$

Quadratic equation in ω^2

$$\omega^2 = \frac{k^2}{2}\left(c_s^2 + V_A^2\right) \pm \frac{k^2}{2}\sqrt{\left(c_s^2 + V_A^2\right)^2 - 4\frac{k_z^2}{k^2}c_s^2V_A^2}$$

The phase speed is obtained by dividing by the wave number.

$$v_{\text{phase}}^2 = \left(\frac{\omega}{k}\right)^2 = \frac{1}{2}\left\{\left(c_s^2 + V_A^2\right) \pm \sqrt{\left(c_s^2 + V_A^2\right)^2 - 4\frac{k_z^2}{k^2}c_s^2V_A^2}\right\}$$

Fast and Slow Modes

$$v_{\text{phase}}^2 = \frac{1}{2}\left\{\left(c_s^2 + V_A^2\right) \pm \sqrt{\left(c_s^2 + V_A^2\right)^2 - 4\frac{k_z^2}{k^2}c_s^2V_A^2}\right\}$$

<u>Magnetosonic Modes</u>

+ sign → *Fast* mode — Faster than both c_s or V_A.

– sign → *Slow* mode — Slower than either c_s or V_A.

The detailed behavior depends on the ratio of c_s and V_A. Traditionally, this is expressed through the plasma parameter β, which is defined as the ratio of the gas pressure to magnetic pressure.

$$\beta \equiv \frac{P}{B^2\,/\,8\pi} = \frac{2}{\gamma}\frac{c_s^2}{V_A^2}$$

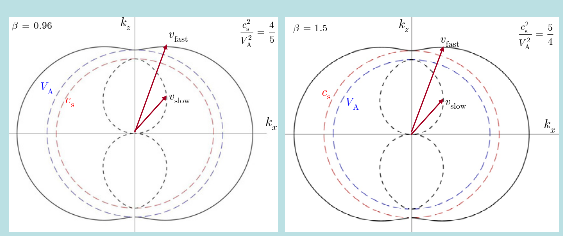
$\beta \ll 1$ Strong field limit

$\beta \gg 1$ Weak field limit

Phase Speed Diagrams

$$v_{\text{phase}}^2 = \frac{1}{2}\left\{\left(c_s^2 + V_A^2\right) \pm \sqrt{\left(c_s^2 + V_A^2\right)^2 - 4\frac{k_z^2}{k^2}c_s^2V_A^2}\right\}$$

The sound speed and the Alfvén speed appear symmetrically



Friedrich's Diagram

Slow and Fast Speeds

$$v_{\rm phase}^2 = \frac{1}{2}\left\{\left(c_{\rm s}^2 + V_{\rm A}^2\right) \pm \sqrt{\left(c_{\rm s}^2 + V_{\rm A}^2\right)^2 - 4\frac{k_z^2}{k^2}c_{\rm s}^2 V_{\rm A}^2}\right\}$$

This equation can be expressed in a useful form using the slow and fast speeds

$$c_{\rm T}^{-2} \equiv c_{\rm s}^{-2} + V_{\rm A}^{-2}$$

Tube Speed / Cusp Speed / Slow Speed

$$c_{\rm T}^2 = \frac{c_{\rm s}^2 V_{\rm A}^2}{c_{\rm s}^2 + V_{\rm A}^2}$$

$$c_{\rm F}^2 \equiv c_{\rm s}^2 + V_{\rm A}^2$$

Fast Speed

$$v_{\rm phase}^2 = \frac{c_{\rm F}^2}{2}\left[1 \pm \sqrt{1 - 4\left(\frac{k_z}{k}\right)^2 \frac{c_{\rm T}^2}{c_{\rm F}^2}}\right]$$

Note: The slow and fast wave do not actually propagate at the slow and fast speeds. Blech!

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Plasma β-parameter

$$v_{\rm phase}^2 = \frac{c_{\rm F}^2}{2}\left[1 \pm \sqrt{1 - 4\left(\frac{k_z}{k}\right)^2 \frac{c_{\rm T}^2}{c_{\rm F}^2}}\right]$$

The tube speed is small if either the sound speed or the Alfvén speed are small compared to the other. This can be expressed through the plasma's β-parameter.

$$c_{\rm T}^2 = \frac{c_{\rm s}^2 V_{\rm A}^2}{c_{\rm s}^2 + V_{\rm A}^2}$$

$$\beta = \frac{8\pi P}{B^2} = \frac{2}{\gamma}\frac{c_{\rm s}^2}{V_{\rm A}^2}$$

$$c_{\rm F}^2 \equiv c_{\rm s}^2 + V_{\rm A}^2$$

If $\beta \ll 1$ then $c_{\rm s}^2 \ll V_{\rm A}^2$, $c_{\rm T}^2 \to c_{\rm s}^2$, $c_{\rm F}^2 \to V_{\rm A}^2$

Strong field limit

$$\frac{c_{\rm T}^2}{c_{\rm F}^2} \to \frac{c_{\rm s}^2}{V_{\rm A}^2} \ll 1$$

small in either limit

If $\beta \gg 1$ then $V_{\rm A}^2 \ll c_{\rm s}^2$, $c_{\rm T}^2 \to V_{\rm A}^2$, $c_{\rm F}^2 \to c_{\rm s}^2$

Weak field limit

$$\frac{c_{\rm T}^2}{c_{\rm F}^2} \to \frac{V_{\rm A}^2}{c_{\rm s}^2} \ll 1$$

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Asymptotic Limits

$$v_{\rm phase}^2 = \frac{c_{\rm F}^2}{2}\left[1 \pm \sqrt{1 - 4\left(\frac{k_z}{k}\right)^2 \frac{c_{\rm T}^2}{c_{\rm F}^2}}\right]$$

If either the sound speed or Alfvén speed are much larger than the other, the square root term may be simplified.

$$v_{\rm phase}^2 = \frac{c_{\rm F}^2}{2}\left[1 \pm \left(1 - 2\left(\frac{k_z}{k}\right)^2 \frac{c_{\rm T}^2}{c_{\rm F}^2} + \cdots\right)\right]$$

$$v_{\rm fast}^2 = c_{\rm F}^2 - \left(\frac{k_z}{k}\right)^2 c_{\rm T}^2 + \cdots \quad \text{Fast Mode}$$

$$v_{\rm slow}^2 = \left(\frac{k_z}{k}\right)^2 c_T^2 + \left(\frac{k_z}{k}\right)^4 \frac{c_T^4}{c_F^2} + \cdots \quad \text{Slow Mode}$$

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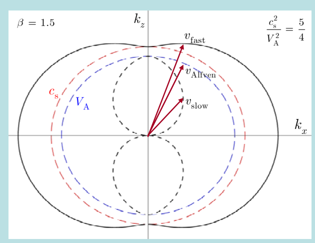
Synopsis

For $c_{\rm s}^2 \ll V_{\rm A}^2$ or $c_{\rm s}^2 \gg V_{\rm A}^2$

(or equivalently $c_{\rm F}^2 \gg c_{\rm T}^2$)



Fast Mode

$$\omega^2 = k^2 c_{\rm F}^2 - k_z^2 c_{\rm T}^2 + \ldots$$

$$v_{\rm phase}^2 = c_{\rm F}^2 - \left(\frac{k_z}{k}\right)^2 c_{\rm T}^2 + \cdots$$

Slow Mode

$$\omega^2 = k_z^2 c_{\rm T}^2 + \frac{k_z^2}{k^2}\frac{c_{\rm T}^2}{c_{\rm F}^2} + \cdots$$

$$v_{\rm phase}^2 = \left(\frac{k_z}{k}\right)^2 c_T^2 + \left(\frac{k_z}{k}\right)^4 \frac{c_T^4}{c_F^2} + \cdots$$

Alfvén Mode

$$\omega^2 = k_z^2 V_{\rm A}^2$$

$$v_{\rm phase}^2 = \left(\frac{k_z}{k}\right)^2 V_{\rm A}^2$$

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Magnetosonic Forces

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Explicit Forms

There are really only two forces at work: gas pressure and the Lorentz force. Gas pressure is aligned with the wavevector and the magnetic forces are perpendicular to the field.

$$P_1 = \frac{\rho_0 c_{\rm s}^2}{\omega}\left(\vec{k}\cdot\vec{v}\right) \quad \Rightarrow \quad -\vec{\nabla}P_1 = -\frac{i\rho_0 c_{\rm s}^2}{\omega}\left(\vec{k}\cdot\vec{v}\right)\vec{k}$$

$$M_1 = \frac{(\vec{\nabla}\times\vec{B}_1)\times\vec{B}_0}{4\pi} \quad \Rightarrow \quad \vec{M}_1 = -\frac{i\rho_0 V_{\rm A}^2}{\omega}k^2 v_x\,\hat{x}$$

Sometimes it's convenient to decompose the magnetic force into a magnetic pressure Π_1 and tension T_1.

$$\Pi_1 = \frac{\rho_0 c_{\rm s}^2}{\omega}k_x v_x \quad \Rightarrow \quad -\vec{\nabla}\Pi_1 = -\frac{i\rho_0 V_{\rm A}^2}{\omega}k_x v_x\,\vec{k}$$

$$\vec{T}_1 = \frac{i\rho_0 V_{\rm A}^2}{\omega}k_z v_x\left(\vec{k}\times\hat{y}\right)$$

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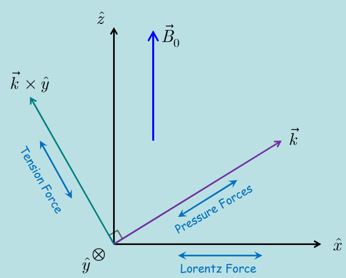
Force Directions



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Fast Mode

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Why is the Fast Mode Fast?

The fast mode is fast because this mode has a polarization that "maximizes" the restoring force.

Weak Field $\frac{c_s^2}{V_A^2} \gg 1 \quad \beta \gg 1$

Gas Pressure Forces are maximized.

$$-\vec{\nabla} P_1 = -\frac{i\rho_0 c_s^2}{\omega}\left(\vec{k}\cdot\vec{v}\right)\vec{k}$$

~Sound Wave

$$\vec{v} = v\hat{k} + \frac{k_x}{k}\frac{V_A^2}{c_s^2}v\,\hat{x} + \cdots$$

$$\omega^2 = k^2 c_s^2 + k_x^2 V_A^2 + \ldots$$

Strong Field $\frac{V_A^2}{c_s^2} \gg 1 \quad \beta \ll 1$

Magnetic Forces are maximized.

$$\vec{M}_1 = -\frac{i\rho_0 V_A^2}{\omega} k^2 v_x\,\hat{x}$$

~Magnetic Wave (Pressure & Tension)

$$\vec{v} = v\hat{x} + \frac{c_s^2}{V_A^2}v\left[\frac{k_x^2 - k_z^2}{k^2}\,\hat{x} + \frac{k_x k_z}{k^2}\,\hat{z}\right] + \cdots$$

$$\omega^2 = k^2 V_A^2 + k_x^2 c_s^2 + \ldots$$

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Fast Mode Speeds

$$\omega^2 = k^2 c_F^2 - k_z^2 c_T^2 + \ldots$$

Asymptotic Dispersion Relation

$$\vec{v}_{\text{phase}} = c_F\hat{k} - \frac{1}{2}\left(\frac{k_z}{k}\right)^2\frac{c_T^2}{c_F}\hat{k} + \cdots$$

Phase speed
$\vec{v}_p \approx c_F\hat{k}$

$c_F^2 \equiv c_s^2 + V_A^2$

$$\vec{v}_{\text{group}} = \vec{v}_{\text{phase}} - \frac{k_z}{k}\frac{c_T^2}{c_F}\hat{z} + \cdots$$

Group speed
$\vec{v}_g \approx c_F\hat{k}$

Properties of the Fast Mode in the Asymptotic Regime $(c_T << c_F)$

- The fast mode propagates at slightly less than the fast speed.
- The group speed points largely in the direction of the wavevector, with a small additional component along the field.

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Slow Mode

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Why is the Slow Mode Slow?

The slow mode is slow because this mode has a polarization that "minimizes" the restoring force.

Weak Field $\frac{c_s^2}{V_A^2} \gg 1 \quad \beta \gg 1$

Gas Pressure Forces are minimized.

$$-\vec{\nabla} P_1 = -\frac{i\rho_0 c_s^2}{\omega}\left(\vec{k}\cdot\vec{v}\right)\vec{k}$$

~Alfvénic Tension Wave

$$\vec{v} = v\,\hat{k}\times\hat{y} + \frac{k_z}{k}\frac{V_A^2}{c_s^2}v\,\hat{x} + \cdots$$

$$\vec{k}\cdot\vec{v} = \frac{k_x k_z}{k}\frac{V_A^2}{c_s^2} + \cdots$$

$$\omega^2 = k_z^2 V_A^2 - \frac{k_x^2 k_z^2}{k^2}\frac{V_A^4}{c_s^2} + \ldots$$

Strong Field $\frac{V_A^2}{c_s^2} \gg 1 \quad \beta \ll 1$

Magnetic Forces are minimized.

$$\vec{M}_1 = -\frac{i\rho_0 V_A^2}{\omega} k^2 v_x\,\hat{x}$$

~Ducted Sound Wave

$$\vec{v} = v\hat{z} - \frac{k_x k_z}{k^2}\frac{c_s^2}{V_A^2}v\cdots$$

$$\omega^2 = k_z^2 c_s^2 - \frac{k_x^2 k_z^2}{k^2}\frac{c_s^4}{V_A^2} + \ldots$$

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Weak Field Limit $\beta \gg 1$

$$\omega^2 = k_z^2 V_A^2 - \frac{k_x^2 k_z^2}{k^2}\frac{V_A^4}{c_s^2} + \dots$$

Dispersion Relation (Eigenvalue)

$$\vec{v} = v\,\hat{k} \times \hat{y} + \frac{k_z}{k}\frac{V_A^2}{c_s^2} v\,\hat{x} + \cdots$$

Polarization (Eigenvector)

The slow mode is a tension wave. Not because the pressures are small compared to the tension, but because the gas and magnetic pressure to leading order cancel each other (equal magnitude and 180 degrees out of phase.

$$-\vec{\nabla} P_1 = -\frac{i\rho_0 c_s^2}{\omega}\left(\vec{k} \cdot \vec{v}\right)\vec{k} \quad \Rightarrow \quad -\frac{i\rho_0 V_A^2}{\omega} k_x k_z v\,\hat{k}$$

$$-\vec{\nabla} \Pi_1 = -\frac{i\rho_0 V_A^2}{\omega} k_x v_x \vec{k} \quad \Rightarrow \quad \frac{i\rho_0 V_A^2}{\omega} k_x k_z v\,\hat{k}$$

$$\vec{T}_1 = \frac{i\rho_0 V_A^2}{\omega} k_z v_x \left(\vec{k} \times \hat{y}\right) \quad \Rightarrow \quad -\frac{i\rho_0 V_A^2}{\omega} k_z^2 v\left(\hat{k} \times \hat{y}\right)$$

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Slow Mode Speeds

$$\omega^2 = k_z^2 c_T^2 + \frac{k_z^4}{k^2}\frac{c_T^4}{c_F^2} + \cdots$$

Asymptotic Dispersion Relation

$$\vec{v}_{\rm phase} = \frac{k_z}{k} c_T \hat{k} + \frac{1}{2}\frac{k_z^2}{k^2}\frac{c_T^2}{c_F^2} + \cdots$$

Phase speed

$$c_T^2 = \frac{c_s^2 V_A^2}{c_s^2 + V_A^2}$$

$$\vec{v}_{\rm group} = c_T \hat{z} + \frac{k_z^2(k^2 + k_x^2)}{k^4}\frac{c_T^3}{c_F^2}\hat{z} - \frac{k_x k_z^3}{k^4}\frac{c_T^2}{c_F^2}\hat{x} + \cdots$$

Group speed

Properties of the Slow Mode in the Asymptotic Regime ($c_T << c_F$)

- The slow mode propagates (phase) slower than the slow speed.
- The slow mode doesn't propagate (group) perpendicular to the field.
- The group speed points largely in the direction of the field, with a magnitude equal to the slow speed.

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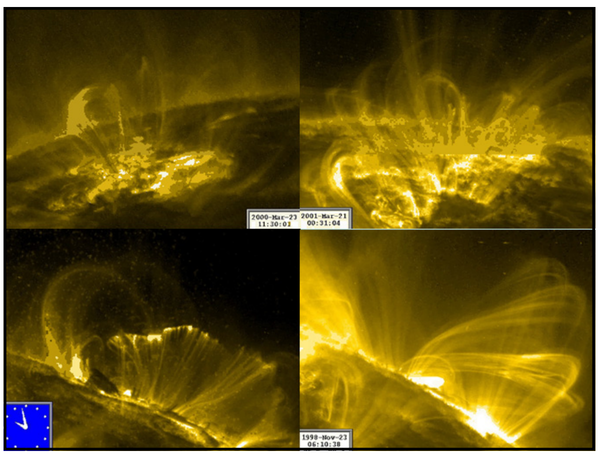

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Bonus Mathematics

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Asymptotics

The following slides slow the path by which one computes the dispersion relations and polarizations in the asymptotic limits of strong and weak field.

I have also include the traditional derivation of the dispersion relations for when the wave propagates purely parallel or purely perpendicular to the magnetic field.

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Asymptotics for the **Fast** Mode for Very Small Slow Speed

Weak Field Limit $\beta \gg 1$

$$\omega^2 = k^2 c_F^2 - k_z^2 c_T^2 + \ldots$$ Dispersion Relation (Eigenvalue)

$$\vec{v} \propto \left(\omega^2 - k_z^2 c_s^2\right)\hat{x} + k_x k_z c_s^2\,\hat{z}$$ Polarization (Eigenvector)

In the limit of weak magnetic field, the fast mode is acoustic in nature with a weak magnetic correction

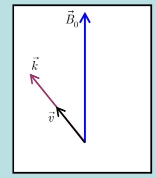
$\beta \gg 1$

$c_s^2 \gg V_A^2$

$$\omega^2 = k^2 c_F^2 - k_z^2 c_T^2 + \ldots$$

$$\omega^2 = k^2\left(c_s^2 + V_A^2\right) - k_z^2 \frac{c_s^2 V_A^2}{c_s^2 + V_A^2} + \ldots$$

$$\omega^2 = k^2 c_s^2 + k_x^2 V_A^2 + \ldots$$



$$\vec{v} \propto \left(\omega^2 - k_z^2 c_s^2\right)\hat{x} + k_x k_z c_s^2\,\hat{z} \Rightarrow \vec{v} = v\left(\hat{k} + \frac{k_x}{k}\frac{V_A^2}{c_s^2}\,\hat{x} + \cdots\right)$$

Longitudinal Motion

Strong Field Limit $\beta \ll 1$

$$\omega^2 = k^2 c_F^2 - k_z^2 c_T^2 + \ldots$$ Dispersion Relation (Eigenvalue)

$$\vec{v} \propto \left(\omega^2 - k_z^2 c_s^2\right)\hat{x} + k_x k_z c_s^2\,\hat{z}$$ Polarization (Eigenvector)

In the limit of strong magnetic field, the fast mode is driven largely by magnetic pressure and tension with a small gas pressure correction.

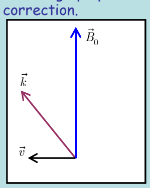
$\beta \ll 1$

$c_s^2 \ll V_A^2$

$$\omega^2 = k^2 c_F^2 - k_z^2 c_T^2 + \ldots$$

$$\omega^2 = k^2\left(V_A^2 + c_s^2\right) - k_z^2 \frac{c_s^2 V_A^2}{c_s^2 + V_A^2} + \ldots$$

$$\omega^2 = k^2 V_A^2 + k_x^2 c_s^2 + \ldots$$



$$\vec{v} \propto \left(\omega^2 - k_z^2 c_s^2\right)\hat{x} + k_x k_z c_s^2\,\hat{z} \Rightarrow \vec{v} = v\left(\hat{x} + \frac{c_s^2}{V_A^2}\left[\frac{k_x^2 - k_z^2}{k^2}\,\hat{x} + \frac{k_x k_z}{k^2}\,\hat{z}\right] + \cdots\right)$$

Transverse Motion (to the field)

Asymptotics for the **Slow** Mode for Very Small Slow Speed

Weak Field Limit $\beta \gg 1$

$$\omega^2 = k_z^2 c_T^2 + \frac{k_z^4}{k^2}\frac{c_T^4}{c_F^2} + \cdots$$ Dispersion Relation (Eigenvalue)

$$\vec{v} = \left(\omega^2 - k_z^2 c_s^2\right)\hat{x} + k_x k_z c_s^2\,\hat{z}$$ Polarization (Eigenvector)

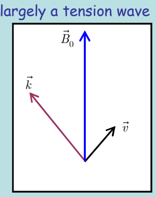
In the limit of weak magnetic field, the slow mode is largely a tension wave and behaves much like the Alfvén wave.

$\beta \gg 1$

$c_s^2 \gg V_A^2$

$$\omega^2 = k_z^2 c_T^2 + \frac{k_z^4}{k^2}\frac{c_T^4}{c_F^2} + \cdots$$

$$\omega^2 = k_z^2 V_A^2 - \frac{k_x^2 k_z^2}{k^2}\frac{V_A^4}{c_s^2} + \cdots$$



$$\vec{v} = \left(\omega^2 - k_z^2 c_s^2\right)\hat{x} + k_x k_z c_s^2\,\hat{z} \Rightarrow \vec{v} = v\left(\hat{k}\times\hat{y} + \frac{k_z}{k}\frac{V_A^2}{c_s^2}\,\hat{x} + \cdots\right)$$

Transverse Motion (to the wavevector)

Weak Field Limit $\beta \gg 1$

$$\omega^2 = k_z^2 V_A^2 - \frac{k_x^2 k_z^2}{k^2}\frac{V_A^4}{c_s^2} + \ldots$$ Dispersion Relation (Eigenvalue)

$$\vec{v} = v\left(\hat{k}\times\hat{y} + \frac{k_z}{k}\frac{V_A^2}{c_s^2}\,\hat{x} + \cdots\right)$$ Polarization (Eigenvector)

The slow mode is a tension wave. Not because the pressures are small compared to the tension, but because the gas and magnetic pressure to leading order cancel each other (equal magnitude and 180 degrees out of phase.

$$-\vec{\nabla}P_1 = -\frac{i\rho_0 c_s^2}{\omega}\left(\vec{k}\cdot\vec{v}\right)\vec{k} \Rightarrow -\frac{i\rho_0 V_A^2}{\omega}k_x k_z u\,\hat{k}$$

$$-\vec{\nabla}\Pi_1 = -\frac{i\rho_0 V_A^2}{\omega}k_x v_x\,\vec{k} \Rightarrow \frac{i\rho_0 V_A^2}{\omega}k_x k_z u\,\hat{k}$$

$$\vec{T}_1 = \frac{i\rho_0 V_A^2}{\omega}k_z v_x\left(\vec{k}\times\hat{y}\right) \Rightarrow -\frac{i\rho_0 V_A^2}{\omega}k_z^2 u\left(\hat{k}\times\hat{y}\right)$$

Strong Field Limit $\beta \ll 1$

$$\omega^2 = k_z^2 c_\mathrm{T}^2 + \frac{k_z^4}{k^2}\frac{c_T^4}{c_F^2} + \cdots$$ Dispersion Relation (Eigenvalue)

$$\vec{v} = \left(\omega^2 - k_z^2 c_\mathrm{s}^2\right)\hat{x} + k_x k_z c_\mathrm{s}^2\ \hat{z}$$ Polarization (Eigenvector)

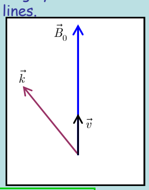
In the limit of strong magnetic field, the slow mode is largely acoustic in nature. However, the wave only propagates along field lines.

$\beta \ll 1$

$c_\mathrm{s}^2 \ll V_\mathrm{A}^2$

$$\omega^2 = k_z^2 c_\mathrm{T}^2 + \frac{k_z^4}{k^2}\frac{c_T^4}{c_F^2} + \cdots$$

$$\omega^2 = k_z^2 c_\mathrm{s}^2 - \frac{k_x^2 k_z^2}{k^2}\frac{c_\mathrm{s}^4}{V_\mathrm{A}^2} + \ldots$$



$$\vec{v} = \left(\omega^2 - k_z^2 c_\mathrm{s}^2\right)\hat{x} + k_x k_z c_\mathrm{s}^2\ \hat{z} \quad \Rightarrow \quad \vec{v} = v\left(\hat{z} - \frac{k_x k_z}{k^2}\frac{c_\mathrm{s}^2}{V_\mathrm{A}^2}\cdots\right)$$

Parallel Motion (to the field) 61

Parallel and Perpendicular Propagation

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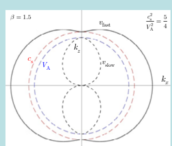
Perpendicular Propagation

If the wave is propagating purely perpendicular to the magnetic field $k_z = 0$

$$\omega^2 = \frac{k^2}{2}\left(c_\mathrm{s}^2 + V_\mathrm{A}^2\right) \pm \frac{k^2}{2}\sqrt{\left(c_\mathrm{s}^2 + V_\mathrm{A}^2\right)^2 - 4\frac{k_z^2}{k^2}c_\mathrm{s}^2 V_\mathrm{A}^2}$$

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}\left(c_\mathrm{s}^2 + V_\mathrm{A}^2\right) \pm \frac{1}{2}\sqrt{\left(c_\mathrm{s}^2 + V_\mathrm{A}^2\right)^2}$$

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}\left(c_\mathrm{s}^2 + V_\mathrm{A}^2\right) \pm \frac{1}{2}\left(c_\mathrm{s}^2 + V_\mathrm{A}^2\right)$$



$$\left(\frac{\omega}{k}\right)^2 = 0 \ \text{ or } \ c_\mathrm{s}^2 + V_\mathrm{A}^2$$

The slow wave doesn't propagate

The fast wave is a magnetosonic pressure wave

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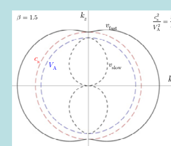
Parallel Propagation

If the wave is propagating purely parallel to the magnetic field $k_x = 0$

$$\omega^2 = \frac{k^2}{2}\left(c_\mathrm{s}^2 + V_\mathrm{A}^2\right) \pm \frac{k^2}{2}\sqrt{\left(c_\mathrm{s}^2 + V_\mathrm{A}^2\right)^2 - 4\frac{k_z^2}{k^2}c_\mathrm{s}^2 V_\mathrm{A}^2}$$

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}\left(c_\mathrm{s}^2 + V_\mathrm{A}^2\right) \pm \frac{1}{2}\sqrt{\left(c_\mathrm{s}^2 + V_\mathrm{A}^2\right)^2 - 4c_\mathrm{s}^2 V_\mathrm{A}^2}$$

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}\left(c_\mathrm{s}^2 + V_\mathrm{A}^2\right) \pm \frac{1}{2}\left(c_\mathrm{s}^2 - V_\mathrm{A}^2\right)$$



$$\left(\frac{\omega}{k}\right)^2 = c_\mathrm{s}^2 \ \text{ or } \ V_\mathrm{A}^2$$

One mode is a sound wave (fast or slow)

The other is identical to the Alfvén Wave (fast or slow)

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