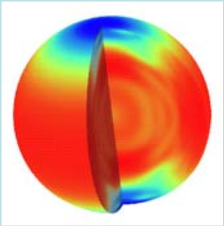
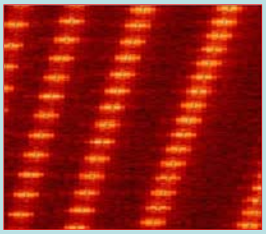
ASTR 7500: Solar & Stellar Magnetism

Hale CGEP Solar & Space Physics





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Lecture 23 Tues 16 Apr 2013

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Lecture 23
Helioseismic Inversions

- Wave Cavities
  - Wave cavities and propagation diagrams
  - Resonances and Eigenfunctions
- Information Content of a Mode's Frequency
  - Eigenproblem
  - Rayleigh quotient
  - Sensitivity Kernels
- Helioseismic Inversions
  - RLS Inversions
  - Regularization

Wave Cavities and Propagation Diagrams

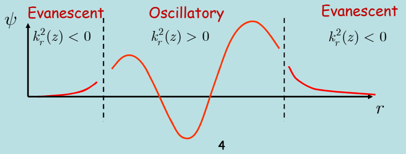
Propagation and Evanescence

$$\frac{d^2\psi}{dr^2} + k_r^2(r)\psi = 0$$

The Helmholtz equation has two different types of behavior:

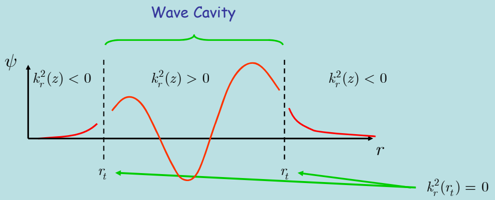
$k_r^2(z) > 0$ **Propagation** – Oscillatory solutions

$k_r^2(z) < 0$ **Evanescence** – Exponentially decaying or growing



Wave Cavities

The waves are largely confined to the region where they propagate. This area is called a "wave cavity".



The boundaries of the wave cavity are where $k_z{}^2(z) = 0$.

Propagation Bands in the Sun

p- and g-Mode Propagation Bands

There are two frequency bands over which waves can propagate. These bands vary as a function of radius,

$$k_r^2(r) = \frac{\omega^2 - \omega_c^2(r)}{c^2(r)} + k_h^2(r)\left(\frac{N^2(r)}{\omega^2} - 1\right)$$

Note that the local dispersion relation is quadratic in ω^2. Hence there are two roots (call them ω_+ and ω_-). We must be able to write the local dispersion relation as follows

$$k_r^2(r) = \frac{1}{\omega^2 c^2(r)}\left[\omega^2 - \omega_+^2(r)\right]\left[\omega^2 - \omega_-^2(r)\right]$$

Propagation occurs when $k_r^2 > 0$. →

$$\left\{\begin{array}{ll} \omega^2 > \omega_-^2 \;\&\; \omega_+^2 & \text{High-Frequency Acoustic Waves} \\ \omega^2 < \omega_-^2 \;\&\; \omega_+^2 & \text{Low-Frequency Internal Gravity Waves} \end{array}\right.$$

Critical Frequencies

The critical frequencies ω_+ and ω_- may be obtained by solving the quadratic equation that results from the dispersion relation when $k_r^2 = 0$.

$$k_r^2(r) = \frac{\omega_\pm^2 - \omega_c^2(r)}{c^2(r)} + k_h^2(r)\left(\frac{N^2(r)}{\omega_\pm^2} - 1\right) = 0$$

$$\boxed{\omega_\pm^2 = \frac{\omega_c^2 + k_h^2 c^2}{2} \pm \sqrt{\left(\frac{\omega_c^2 + k_h^2 c^2}{2}\right)^2 - k_h^2 c^2 N^2}}$$

Lamb Frequency

$$\omega_L^2 \equiv k_h^2 c^2$$

If N^2 is very small compared to $\omega_L^2 = k_h^2 c^2$

$$\omega_+^2 \approx \omega_c^2 + k_h^2 c^2 + \mathcal{O}\left(N^2 / k_h^2 c^2\right)$$

$$\omega_-^2 \approx N^2 + \mathcal{O}\left(N^4 / k_h^4 c^4\right)$$

Propagation if

$\omega^2 > \omega_c^2 + \omega_L^2$ Acoustic Waves

$\omega^2 < N^2$ Internal Gravity Waves

Sir Horace Lamb (1849-1934)

Lamb was a British applied mathematician who authored several influential books on classical physics (still in print)

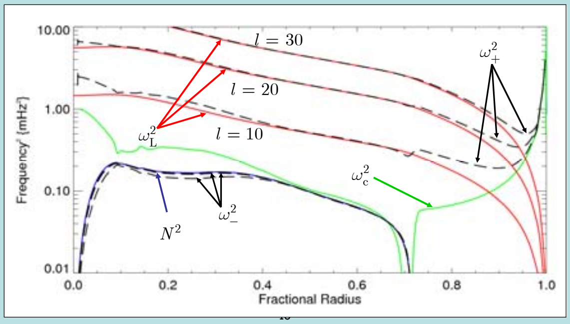
Hydrodynamics (1879)

Dynamical Theory of Sound (1910)

"I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic."
- Sir Horace Lamb, 1932

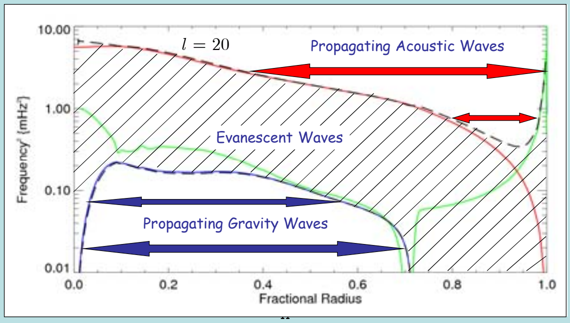
ω_+ and ω_- within the Sun

Lamb Frequency $\omega_L^2 \equiv k_h^2 c^2 = \dfrac{l(l+1)c^2(r)}{r^2}$



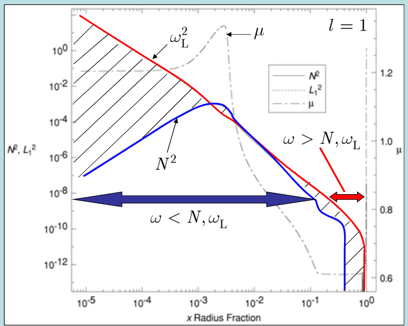
Zones of Propagation

The boundaries of the wave cavities are specified by $\omega^2 = \omega_\pm^2$



Propagation Bands In Evolved Stars

Red Giant – α UMa



$4.25\ \mathrm{M}_\odot$

Burnt He core

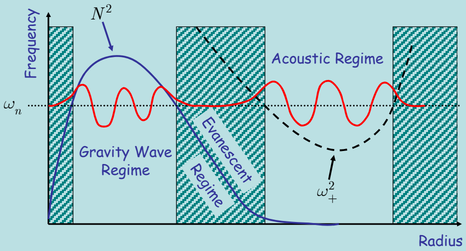
H burning shell

Deep convective envelope

Mixed modes

$\mu =$ mean molecular weight

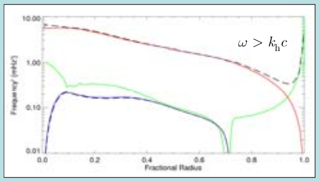
Mixed Modes



This is a schematic example of a mode that has both p-mode and g-mode characteristics. It has oscillations in both cavities.

Limiting Dispersion Relations

High-Frequency Limit (*p* modes)



For $l > 10$

$$N \ll \omega_L = k_h c$$

Thus for <u>acoustic waves</u> we can ignore the buoyancy frequency.

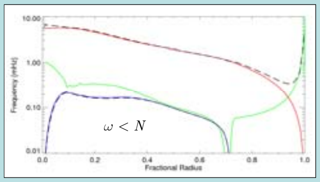
Set the buoyancy frequency to zero in the local dispersion relation

$$k_r^2(r) = \frac{\omega^2 - \omega_c^2(r)}{c^2(r)} + k_h^2(r)\left(\cancel{\frac{N^2(r)}{\omega^2}} - 1\right)$$

Total wavenumber
$k^2 = k_h^2 + k_z^2$

$$k_r^2(r) \approx \frac{\omega^2 - \omega_c^2(r)}{c^2(r)} - k_h^2(r) \quad \Longrightarrow \quad \boxed{\omega^2 \approx k^2(r)c^2(r) + \omega_c^2(r)}$$

Low-Frequency Limit (*g* modes)



For $l > 10$

$$N \ll \omega_L = k_h c$$

For <u>internal gravity waves</u> we can ignore terms involving the inverse sound speed.

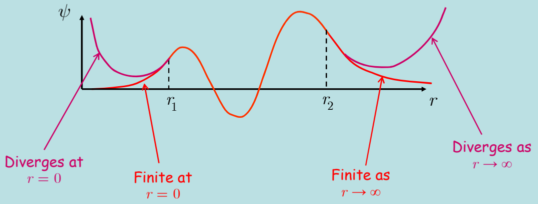
Set the sound speed to infinity in the local dispersion relation

$$k_r^2(r) = \cancel{\frac{\omega^2 - \omega_c^2(r)}{c^2(r)}} + k_h^2(r)\left(\frac{N^2(r)}{\omega^2} - 1\right)$$

$$\boxed{k_r^2(r) \approx k_h^2(r)\left(\frac{N^2(r)}{\omega^2} - 1\right)} \quad \Longrightarrow \quad \boxed{\omega^2 \approx \frac{k_h^2(r)}{k^2(r)} N^2(r)}$$

Resonances

Resonance Condition



Diverges at $r = 0$

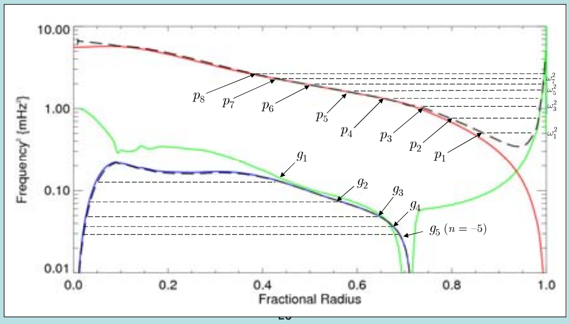
Finite at $r = 0$

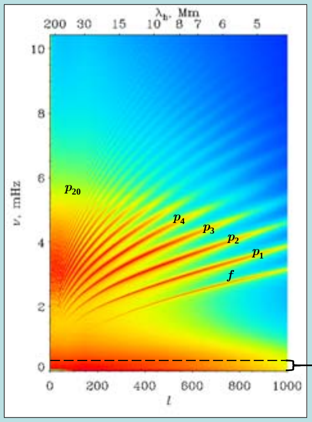
Finite as $r \to \infty$

Diverges as $r \to \infty$

The radial resonance condition is that the solutions is finite at the center of the star and that the energy is finite. This last requirement requires that the solution decays (instead of grows) with height above the photosphere.

Frequency Sequence





The sequence of p modes is observed in the sun.

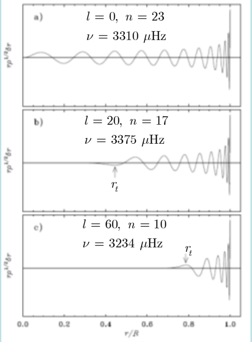
The g modes would be very low frequency and have yet to be observed.

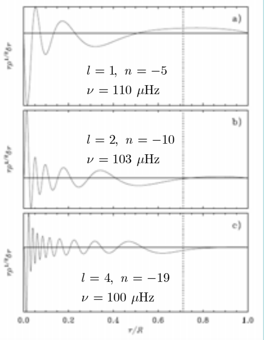
g modes?

Eigenfunctions

p modes

g modes





Information Content of a Mode Frequency

Eigenvalue Equation

Remember last time I showed that the sun's oscillations obeyed a Helmholtz equation in radius , and given boundary conditions formed an eigenvalue-eigenfunction problem.

$$\frac{d^2\psi}{dr^2} + k_r^2(r)\ \psi(r) = 0$$

$$k_r^2(r) = \frac{\omega^2 - \omega_c^2(r)}{c^2(r)} + k_h^2(r)\left(\frac{N^2(r)}{\omega^2} - 1\right)$$

If we take the high-frequency limit (p modes), then we can express this differential equation in a standard eigenproblem form

$$\frac{d^2\psi}{dr^2} + \left[\frac{\omega^2 - \omega_c^2(r)}{c^2(r)} - k_h^2(r)\right]\psi(r) = 0$$

$$\frac{d^2\psi}{dr^2}+\left[\frac{\omega^2-\omega_{\mathrm{c}}^2(r)}{c^2(r)}-k_{\mathrm{h}}^2(r)\right]\psi(r)=0$$

Multiply through by the square of the sound speed and rearrange

$$\left\{c^2(r)\frac{d^2}{dr^2}-\left[\omega_{\mathrm{c}}^2(r)+k_{\mathrm{h}}^2(r)c^2(r)\right]\right\}\psi(r)+\omega^2\psi(r)=0$$

Defining the differential operator $\boldsymbol{\mathcal{L}}$,

$$\boldsymbol{\mathcal{L}}\equiv\left\{c^2\frac{d^2}{dr^2}-\left(\omega_{\mathrm{c}}^2+k_{\mathrm{h}}^2c^2\right)\right\}$$

Note:
$\mathcal{L}$ is a differential operator that depends on the sound speed and the acoustic cutoff frequency (or density).

we can write this ODE in the form of a standard eigenvalue equation.

$$\boldsymbol{\mathcal{L}}\psi+\omega^2\psi=0$$

or

$$\boxed{\boldsymbol{\mathcal{L}}\psi_{nl}+\omega_{nl}^2\psi_{nl}=0}$$

Sturm-Liouville System

$$\boldsymbol{\mathcal{L}}\psi_n+\omega_n^2\psi_n=0$$

Drop l index for notational simplicity.

The operator $\boldsymbol{\mathcal{L}}$ is a Sturm-Liouville operator. Therefore, we immediately know that its eigenfunction are orthogonal with a weight function. The orthogonality integral is over the entire interior of the star, from its center $r = 0$ to its surface $r = R$.

$$\int_0^R W(r)\,\psi_n(r)\psi_p(r)\ dr=\delta_{np}$$

Integrate over the entire star

$$W(r)=\frac{R}{Mg(R)}\ \frac{1}{c^2(r)}$$

Allows the normalized eigenfunctions to retain their physical units.

M = Stellar Mass

R = Stellar Radius

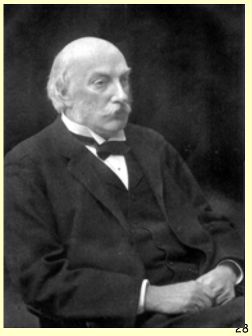
$g(R)$ = Surface gravity

What use are the Eigenfrequencies?

The eigenfrequencies are a weighted spatial average of the interior properties of the star. To see this we will compute the Rayleigh Quotient.

The Rayleigh Quotient is a useful quantity that allows one to prove all sorts of mathematical niceties about the eigenvalues of a differential equation.

John William Strutt (1842-1919) aka Lord Rayleigh



Lord Rayleigh had his fingers in everything physical and mathematical. His name appears over and over again in a variety of contexts.

British Physicist

- Acoustics (Sound localization in human hearing, 1900)
- Optics (Rayleigh Criterion, Rayleigh-Jeans law 1900, Rayleigh scattering – why is the sky blue?)
- Ferromagnetism (Rayleigh Law 1887)
- Chemistry (discovered the element argon with Ramsay, Nobel Prize 1904)
- Hydrodynamics (Rayleigh-Bénard convection 1916, Rotating Couette flows 1880 & 1916, Rayleigh-Taylor instability 1883)

Computing the Rayleigh Quotient

To compute the Rayleigh Quotient, we multiple the ODE by an eigenfunction and the weight function and then integrate over the domain (if you know what this means, we take the inner product in Hilbert space of the ODE with an eigenfunction). Then solve for the eigenvalue.

$$\boldsymbol{\mathcal{L}}\psi_n+\omega_n^2\psi_n=0$$

For convenience, drop the horizontal quantum number l

$$\int_0^R W\psi_n\left[\boldsymbol{\mathcal{L}}\psi_n+\omega_n^2\psi_n\right]dr=0$$

Integrate over the entire star

Rearrange terms

$$\omega_n^2\int_0^R W\psi_n^2\ dr=-\int_0^R W\psi_n\boldsymbol{\mathcal{L}}\psi_n\ dr$$

$$\omega_n^2\int_0^R W\psi_n^2\ dr=-\int_0^R W\psi_n\boldsymbol{\mathcal{L}}\psi_n\ dr$$

The integral on the left-hand side is our orthogonality integral $\int_0^R W\psi_n\psi_p\ dr=\delta_{np}$

$$\omega_n^2=-\int_0^R W\psi_n\boldsymbol{\mathcal{L}}\psi_n\ dr$$

$$\boldsymbol{\mathcal{L}}\equiv\left\{c^2\frac{d^2}{dr^2}-\left(\omega_{\mathrm{c}}^2+k_{\mathrm{h}}^2c^2\right)\right\}$$

$$=-\int_0^R W\psi_n\overbrace{\left\{c^2\frac{d^2}{dr^2}-\left(\omega_{\mathrm{c}}^2+k_{\mathrm{h}}^2c^2\right)\right\}}\psi_n\ dr$$

Therefore,

$$\boxed{\omega_n^2=-\int_0^R\left[c^2W\psi_n\left(\psi_n''-k_{\mathrm{h}}^2\psi_n\right)-\omega_{\mathrm{c}}^2W\psi_n^2\right]dr}$$

Sensitivity Kernels

$$\omega_n^2 = -\int_0^R \left[W\psi_n c^2 \left(\psi_n'' - k_h^2 \psi_n \right) - \omega_c^2 W \psi_n^2 \right] dr$$

This can be rewritten in a more illustrative form by defining kernel functions.

$$\boxed{\omega_n^2 = \int_0^R \left[c^2(r)\mathcal{C}_n(r) + \omega_c^2(r)\mathcal{W}_n(r) \right] dr}$$

$\mathcal{C}_n(r) \equiv -W\left(\psi_n \psi_n'' - k_h^2 \psi_n^2 \right)$ Kernel for the sound speed

$\mathcal{W}_n(r) \equiv W\psi_n^2$ Kernel for the cut-off frequency

The frequencies are a weighted average of the properties of the stellar interior. The weights depend on the eigenfunctions. Therefore, the frequencies are an average of the stellar properties only within the acoustic cavity for that mode.

Perturbative Techniques

The Rayleigh Quotient is rarely used directly in practice. When trying to use measured frequencies to deduce the internal properties of a star, the eigenfunctions (and therefore the kernels) aren't known until you have already solved the problem. Because of this difficulty, perturbation theory is usually employed.

Assume that a stellar structure aficionado has provided us with a reference stellar model, which does a good job of predicting the eigenfrequencies (it's a good model but not perfect).

Reference model provides both atmospheric profiles and modes

$c^2(r) = \tilde{c}^2(r)$

$\omega_c^2(r) = \tilde{\omega}_c^2(r)$

The tilde indicates the reference model

$$\boxed{\begin{gathered} \tilde{\mathcal{L}}\tilde{\psi}_n + \tilde{\omega}_n^2 \tilde{\psi}_n = 0 \\ \int_0^R W\tilde{\psi}_n \tilde{\psi}_p \; dr = \delta_{np} \end{gathered}}$$

Perturbations

Assume that the true stellar profiles of sound speed and cutoff frequency are small perturbations from the reference model (remember that the reference model is a good model).

$$c^2(r) = \tilde{c}^2(r) + \delta c^2(r)$$

$$\omega_c^2(r) = \tilde{\omega}_c^2(r) + \delta\omega_c^2(r)$$

We can then safely assume that the true eigenfunctions and eigenfrequencies can be treated perturbatively as well.

$$\omega_n^2 = \tilde{\omega}_n^2 + \delta\omega_n^2$$

$$\psi_n(r) = \tilde{\psi}_n(r) + \delta\psi_n(r)$$

$$\mathcal{L} = \tilde{\mathcal{L}} + \delta\mathcal{L}$$

Note:

The eigenfrequencies and eigenfunctions for the reference model are known.

The true eigenfrequencies (and hence the perturbation to the frequencies) can be measured.

What isn't known is the atmospheric perturbations δc^2 and $\delta\omega_c^2$.

Sensitivity Kernels

Using standard perturbation theory, where one assumes that differences between the reference model and the real star are small, one can derive† the following integral equation

$$\boxed{\delta\omega_n^2 = \int_0^R \left[\mathcal{C}_n(r)\frac{\delta c^2}{\tilde{c}^2} + \mathcal{W}_n(r)\frac{\delta\omega_c^2}{\tilde{\omega}_c^2} \right] dr}$$

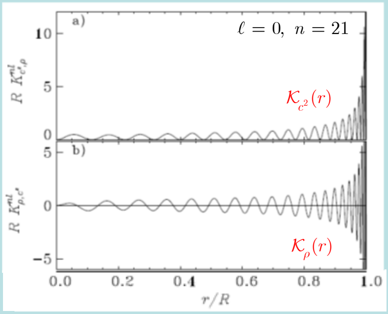
$\tilde{\mathcal{C}}_n(r) \equiv -\tilde{c}^2\; W\left(\tilde{\psi}_n'' \tilde{\psi}_n - k_h^2 \tilde{\psi}_n^2 \right)$ ← Sensitivity Kernel for the fractional perturbation in the sound speed

$\mathcal{W}_n(r) \equiv \tilde{\omega}_c^2\; W\,\tilde{\psi}_n^2$ ← Sensitivity Kernel for the fractional perturbation in the cutoff frequency

† See the end of this lecture for a full derivation

Mode Kernels

$$\delta\omega_n^2 = \int \left(\mathcal{K}_{c^2}(r)\frac{\delta c^2}{\tilde{c}^2} + \mathcal{K}_\rho(r)\frac{\delta\rho}{\tilde{\rho}} \right) dr$$



Positive

Positive & Negative

These are kernels for sound speed (squared) and density

Helioseismic Inversions

Coupled Integral Equations

Consider a case where we wish to ignore the effects of ionization. (I'm doing this solely for simplicity of argument.) In such a situation, the density perturbation is linearly related to the sound speed perturbation. Frequency observations can then be characterized as follows

$$\delta\omega_{ln}^2 = \int \mathcal{K}_{ln}(r)\frac{\delta c^2}{\tilde{c}^2}\,dr \pm \sigma_{ln}$$

Observational Data

Sensitivity Kernels (Reference Model)

Unknown!

Observational Error

Our goal is to invert this set of coupled integral equations to obtain the internal structure.

Regularized Least Squares (RLS) Inversion

We try to describe the internal sound speed perturbation with a parameterization.

$$\frac{\delta c^2}{\tilde{c}^2} = f\left(r;\boldsymbol{\alpha}\right) \qquad \boldsymbol{\alpha} = \left\{\alpha_j\right\}$$

The α_j are the model's free parameters and the parameterization could be an expansion in a set of basis functions (such as sines, Chebyshev Polynomials, etc.).

$$f\left(r\right) = \sum_j \alpha_j T_j(r)$$

The parameters could also be the value of the sound speed perturbation δc^2 at a predetermined grid of radii r_j. The perturbed sound speed at radii in between grid points might be defined through interpolation (spline, bilinear, etc.).

$$\alpha_j = \frac{\delta c^2(r_j)}{\tilde{c}^2(r_j)} = f\left(r_j\right)$$

$$f(r) = \mathrm{SPLINE}(r_j, \alpha_j; r)$$

Goodness of Fit

In an RLS inversion the parameters α_j are chosen to minimize the goodness of fit (in a least square sense).

Sum over Measured Modes

Parameters (Unknown)

$$\chi^2(\boldsymbol{\alpha}) = \sum_{nl}\frac{1}{\sigma_{nl}^2}\left[\delta\omega_{nl}^2 - \int_0^R \mathcal{K}_{nl}(r)\,f\left(r;\boldsymbol{\alpha}\right)dr\right]^2$$

Estimate of the measurement errors

Data (Observed)

Sensitivity Kernel (Reference Model)

Fitting function

Regularization

The solution obtained is not ensured in any way of being smooth. One could (and will) get a solution where f possesses many extreme wiggles with radius. The way to rectify this problem is to include a term in the goodness of fit that penalizes wiggly functions. This is called regularization. One can choose any penalty (or regularization) term that accomplishes this, but in practice one often takes

$$\chi^2(\boldsymbol{\alpha}) = \sum_{nl}\frac{1}{\sigma_{nl}^2}\left[\delta\omega_{nl}^2 - \int_0^R \mathcal{K}_{nl}(r)\,f\left(r;\boldsymbol{\alpha}\right)dr\right]^2 + \lambda\int_0^R\left(\frac{\partial f}{\partial r}\right)^2 dr$$

Trade-off Parameter

The trade-off parameter λ can be freely chosen. The higher its value, the smoother the solution. However, the higher the value, the higher the error in the solutions as well (hence trade-off).

Bonus Feature Perturbation Theory

Perturbation Theory

What follows is a demonstration of how perturbation theory can be used to calculate sensitivity kernels for a given stellar reference model.

Reference Model

The Rayleigh Quotient is rarely used directly in practice. When trying to use measured frequencies to deduce the internal properties of a star, the eigenfunctions (and therefore the kernels) aren't known until you have already solved the problem. Because of this difficulty, perturbation theory is usually employed.

Assume that a stellar structure aficionado has provided us with a reference stellar model, which does a good job of predicting the eigenfrequencies (it's a good model but not perfect).

<u>Reference model provides both atmospheric profiles and modes</u>

$$c^2(r) = \tilde{c}^2(r)$$

$$\omega_{\mathrm{c}}^2(r) = \tilde{\omega}_{\mathrm{c}}^2(r)$$

The tilde indicates the reference model

The eigenfunctions of the reference model form a complete orthogonal set.

$$\boxed{\begin{gathered}\tilde{\boldsymbol{\mathcal{L}}}\tilde{\psi}_n + \tilde{\omega}_n^2\tilde{\psi}_n = 0 \\ \int_0^R W\tilde{\psi}_n\tilde{\psi}_p \; dr = \delta_{np}\end{gathered}}$$

Perturbations

Assume that the true stellar profiles of sound speed and cutoff frequency are small perturbations from the reference model (remember that the reference model is a good model).

$$c^2(r) = \tilde{c}^2(r) + \delta c^2(r)$$

$$\omega_{\mathrm{c}}^2(r) = \tilde{\omega}_{\mathrm{c}}^2(r) + \delta\omega_{\mathrm{c}}^2(r)$$

We can then safely assume that the true eigenfunctions and eigenfrequencies can be treated perturbatively as well.

$$\omega_n^2 = \tilde{\omega}_n^2 + \delta\omega_n^2$$

$$\psi_n(r) = \tilde{\psi}_n(r) + \delta\psi_n(r)$$

$$\boldsymbol{\mathcal{L}} = \tilde{\boldsymbol{\mathcal{L}}} + \delta\boldsymbol{\mathcal{L}}$$

<u>Note:</u>
The eigenfrequencies and eigenfunctions for the reference model are <u>known</u>.

The true eigenfrequencies (and hence the perturbation to the frequencies) can be <u>measured</u>.

What isn't known is the atmospheric perturbations δc^2 and $\delta\omega_c^2$.

Perturb the Operator

Remember the operator depends on the sound speed and acoustic cutoff frequency.

$$\begin{tikzcd} \boldsymbol{\mathcal{L}} = \left\{ c^2 \frac{d^2}{dr^2} - \left(\omega_{\mathrm{c}}^2 + k_{\mathrm{h}}^2 c^2 \right) \right\} \arrow[r] \arrow[dr] & \tilde{\boldsymbol{\mathcal{L}}} = \left\{ \tilde{c}^2 \frac{d^2}{dr^2} - \left(\tilde{\omega}_{\mathrm{c}}^2 + k_{\mathrm{h}}^2 \tilde{c}^2 \right) \right\} \\ & \delta\boldsymbol{\mathcal{L}} = \left\{ \delta c^2 \frac{d^2}{dr^2} - \left(\delta\omega_{\mathrm{c}}^2 + k_{\mathrm{h}}^2 \delta c^2 \right) \right\} \end{tikzcd}$$

$$\boxed{\delta\boldsymbol{\mathcal{L}} = \delta c^2 \left(\frac{d^2}{dr^2} - k_{\mathrm{h}}^2 \right) - \delta\omega_{\mathrm{c}}^2} \longleftarrow$$

The perturbed operator depends linearly on the perturbed thermodynamic variables (unknowns)

Perturbation Equation

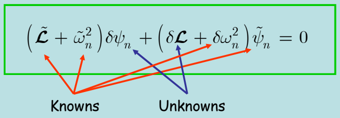
Perturb the wave equation and keep only the terms that are linear in the perturbation

$$\left(\boldsymbol{\mathcal{L}} + \omega_n^2 \right) \psi_n = 0$$

<u>Reference Model Eigenproblem</u>

$$\left(\tilde{\boldsymbol{\mathcal{L}}} + \tilde{\omega}_n^2 \right) \tilde{\psi}_n = 0$$

<u>Linearized ODE for perturbations</u>



$$\left(\tilde{\boldsymbol{\mathcal{L}}} + \tilde{\omega}_n^2 \right) \delta\psi_n + \left(\delta\boldsymbol{\mathcal{L}} + \delta\omega_n^2 \right) \tilde{\psi}_n = 0$$

Remember, that the perturbed operator contains the atmospheric perturbations that we would like to deduce.

$$\delta\boldsymbol{\mathcal{L}} = \delta c^2 \left(\frac{d^2}{dr^2} - k_{\mathrm{h}}^2 \right) - \delta\omega_{\mathrm{c}}^2$$

Expansion in Reference Eigenfunctions

The eigenfunctions of the reference model form a complete orthogonal set. Therefore, we can represent any reasonable function as a linear combination of eigenfunctions.

Expand the perturbed eigenfunctions in the eigenfunctions of the reference model

$$\delta\psi_n(r) = \sum_p A_{np} \tilde{\psi}_p(r)$$

Insert this expansion into the ODE for the perturbations

$$\left(\tilde{\boldsymbol{\mathcal{L}}} + \tilde{\omega}_n^2 \right) \delta\psi_n + \left(\delta\boldsymbol{\mathcal{L}} + \delta\omega_n^2 \right) \tilde{\psi}_n = 0$$

$$\left(\tilde{\boldsymbol{\mathcal{L}}} + \tilde{\omega}_n^2 \right) \sum_p A_{np} \tilde{\psi}_p = -\left(\delta\boldsymbol{\mathcal{L}} + \delta\omega_n^2 \right) \tilde{\psi}_n$$

$$\left(\tilde{\boldsymbol{\mathcal{L}}} + \tilde{\omega}_n^2\right)\sum_p A_{np}\tilde{\psi}_p = -\left(\delta\boldsymbol{\mathcal{L}} + \delta\omega_n^2\right)\tilde{\psi}_n$$

Transfer the operator on the left inside the summation

$$\sum_p A_{np}\left(\tilde{\boldsymbol{\mathcal{L}}} + \tilde{\omega}_n^2\right)\tilde{\psi}_p = -\left(\delta\boldsymbol{\mathcal{L}} + \delta\omega_n^2\right)\tilde{\psi}_n$$

Use the eigenvalue equation for the reference model $\tilde{\boldsymbol{\mathcal{L}}}\tilde{\psi}_n = -\tilde{\omega}_n^2\ \tilde{\psi}_n$

$$\sum_p A_{np}\left(-\tilde{\omega}_p^2 + \tilde{\omega}_n^2\right)\tilde{\psi}_p = -\left(\delta\boldsymbol{\mathcal{L}} + \delta\omega_n^2\right)\tilde{\psi}_n$$

$$\boxed{\sum_p A_{np}\left(\tilde{\omega}_n^2 - \tilde{\omega}_p^2\right)\tilde{\psi}_p = -\left(\delta\boldsymbol{\mathcal{L}} + \delta\omega_n^2\right)\tilde{\psi}_n}$$

Inner Product in Hilbert Space

$$\sum_p A_{np}\left(\tilde{\omega}_n^2 - \tilde{\omega}_p^2\right)\tilde{\psi}_p = -\left(\delta\boldsymbol{\mathcal{L}} + \delta\omega_n^2\right)\tilde{\psi}_n$$

Multiply by $W\tilde{\psi}_n$ and integrate over the entire star

$$\sum_p A_{np}\left(\tilde{\omega}_n^2 - \tilde{\omega}_p^2\right)\int_0^R W\tilde{\psi}_p\tilde{\psi}_n\ dr = -\int_0^R W\tilde{\psi}_n\left(\delta\boldsymbol{\mathcal{L}} + \delta\omega_n^2\right)\tilde{\psi}_n\ dr$$

Use the orthonormality of the eigenfunctions of the reference model to eliminate the LHS and rewrite the term with perturbed frequency. $\int_0^R W\tilde{\psi}_n\tilde{\psi}_p\ dr = \delta_{np}$

$$\boxed{\delta\omega_n^2 = -\int_0^R W\tilde{\psi}_n\delta\boldsymbol{\mathcal{L}}\tilde{\psi}_n\ dr}$$

Perturbed Frequency

$$\delta\omega_n^2 = -\int_0^R W\tilde{\psi}_n\delta\boldsymbol{\mathcal{L}}\tilde{\psi}_n\ dr$$

In the parlance of quantum mechanics, the perturbed frequency is proportional to the diagonal matrix elements of the perturbed operator

$$\delta\omega_n^2 = -\left\langle\tilde{\psi}_n\middle|\delta\boldsymbol{\mathcal{L}}\middle|\tilde{\psi}_n\right\rangle = -\delta\boldsymbol{\mathcal{L}}_n^n$$

Remember our previous derivation of the perturbed operator $\delta\boldsymbol{\mathcal{L}} = \delta c^2\left(\frac{d^2}{dr^2} - k_{\mathrm{h}}^2\right) - \delta\omega_{\mathrm{c}}^2$

$$\boxed{\delta\omega_n^2 = -\int_0^R W\tilde{\psi}_n\left[\delta c^2\left(\frac{d^2}{dr^2} - k_{\mathrm{h}}^2\right)\tilde{\psi}_n - \delta\omega_{\mathrm{c}}^2\tilde{\psi}_n\right]\ dr}$$

Sensitivity Kernels

$$\delta\omega_n^2 = -\int_0^R W\tilde{\psi}_n\left[\delta c^2\left(\frac{d^2}{dr^2} - k_{\mathrm{h}}^2\right)\tilde{\psi}_n - \delta\omega_{\mathrm{c}}^2\tilde{\psi}_n\right]\ dr$$

We can now define sensitivity kernels for the fractional perturbations in the atmospheric profiles.

$$\boxed{\delta\omega_n^2 = \int_0^R\left[\mathcal{C}_n(r)\frac{\delta c^2}{\tilde{c}^2} + \mathcal{W}_n(r)\frac{\delta\omega_{\mathrm{c}}^2}{\tilde{\omega}_{\mathrm{c}}^2}\right]dr}$$

$$\mathcal{C}_n(r) \equiv -\tilde{c}^2\ W\tilde{\psi}_n\left(\tilde{\psi}_n'' - k_{\mathrm{h}}^2\tilde{\psi}_n\right) \longleftarrow$$

Sensitivity Kernel for the fractional perturbation in the sound speed

$$\mathcal{W}_n(r) \equiv \tilde{\omega}_{\mathrm{c}}^2\ W\tilde{\psi}_n^2 \longleftarrow$$

Sensitivity Kernel for the fractional perturbation in the cutoff frequency